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STUDY OF COSMIC RAYS
IN THE
SOLAR ENVIRONMENT

by

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16. Abstract

We formulate the problem of the transport of cosmic rays in full generality from first principles (Chapter II). A master equation is deduced which allows for a closed formulation of the transport problem. We show that two dimensionless parameters (a measure of particle energy and a measure of field fluctuation strength) characterize the transport regimes. We show that previous modulation theories are incorrect except for high energies ($\gtrsim$ 1 BeV) where solar modulation effects are observed to be very small. The main outcome of our effort is a new convection-diffusion equation which is valid in the energy region where solar modulation is observed.

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PREFACE

The main results of our analysis of the transport of cosmic rays in the interplanetary and interstellar media are studied and summarized. We show that previous modulation theories are incorrect except for high energies ($\gtrsim$ 1 BeV) where solar modulation effects are observed to be very small. The main outcome of our effort is a new convection-diffusion equation which is valid in the energy region where solar modulation is observed.

We formulate the problem of the transport of cosmic rays in full generality from first principles (Chapter II). A master equation is deduced which allows for a closed formulation of the transport problem. We show that two dimensionless parameters (a measure of particle energy and a measure of field fluctuation strength) characterize the transport regimes. In the ultrahigh energy regime where the Fokker-Planck equation holds, contact is established with conventional theory. We then resum the set of contributions that corresponds to only two-point correlations with straight line trajectories. This resummed equation allows us to extrapolate from ultrahigh to high energies.

We then deduce the equations for the omnidirectional intensity and flux as the first two spherical harmonic components of the velocity distribution function (Chapter III). These equations are the basis for transport theory in the modulation energy range. To discuss these equations, we develop suitable asymptotic expansions (Chapters IV, V) which are applied to deduce the cosmic ray transport theory.

We strongly recommend that our transport theory be applied to the wide range of modulation problems of current interest since it is the only <u>derived</u> theory which is applicable in the modulation region.

A summary of special problems and publications is given on page x immediately preceding the main body of this report.

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LIST OF ABBREVIATIONS AND SYMBOLS

A	= finite-order matrix operator
$A(r)$	= two-point correlation coefficient
$A_\ell(\lambda)$	= ℓ'th-order coefficient in Legendre polynomial expansion of $A(r)$
a_m	= m'th eigenvalue of A
B	= finite-order matrix operator
$B_{p\mu,n\beta}$	= coefficient in eigenvector expansion of B
$\vec{B}$	= total magnetic field
$\langle\vec{B}\rangle$	= mean magnetic field
$\vec{B}'$	= random magnetic field
$B(r)$	= two-point correlation function coefficient
$B_\ell(\lambda)$	= ℓ'th-order coefficient of expansion of $B(r)$ in Legendre polynomials
$\tilde{B}$	= rotation of operator B
C	= finite-order matrix operator
C^+	= transpose of C
C^m_β	= matrix element of C
c	= speed of light
$E_1(\omega)$	= first-order exponential integral
$E_2(z)$	= second-order exponential integral
f	= phase space distribution function
$\underline{f}_0, \underline{f}_1, \underline{f}_2, \underline{f}_{20}, \underline{f}_{21}, \underline{f}_{22}$	= expansion terms of f
$\bar{f}$	= extension of f
$f_{\ell m}$	= expansion coefficients in spherical harmonic expansion of f

$\tilde{f}(\omega)$	= Laplace transform of f
$\bar{f}, \bar{f}_0, \bar{f}_1$	= expansion terms of $\bar{f}$
G_0	= Green's integral operator for mean Hamiltonian operator
G	= Green's integral operator for total Hamiltonian operator
G_{00}	= Green's integral operator for mean Hamiltonian operator with no spatial gradients
g	= interaction representation of f
I	= omnidirectional intensity
$J_{\perp}$	= angular average of two-point correlation coefficient
$J_{\parallel}$	= angular average of two-point correlation coefficient
K	= generator of field-free particle motion
$K^{(1)}, K^{(2)}$	= angular averages of two-point correlation function
$\|m\rangle$	= nondegenerate eigenfunction of A
$\|m\alpha\rangle$	= degenerate eigenfunction of A
$\langle m\alpha\|$	= transpose of $\|m\alpha\rangle$
$\|m\alpha\rangle^*$	= complex conjugate of $\|m\alpha\rangle$
N	= perpendicular projection operator relative to $\hat{\beta}$
n_{ij}	= perpendicular projection operator in momentum space
P	= parallel projection operator relative to $\hat{\beta}$
$P_{\perp}$	= normal power spectral coefficient
$\hat{p}$	= unit momentum vector
$\hat{p}_i$	= i'th component of the unit momentum vector

p_k	= k'th component of the momentum vector
R_{ij}	= two-point correlation tensor
r	= magnitude of $\vec{v}$
r_g	= gyroradius in mean magnetic field
$\vec{r}$	= particle separation during interaction with random field
U	= unitary matrix operator
U_0, U_1, U_2	= expansion terms in U
$u_\perp$	= world velocity projected by $\prod_\perp$
u^1, u^2	= world velocities projected by $\prod^{(1)}$ and $\prod^{(2)}$
v	= particle speed
$v_\perp$	= particle speed, projected by $\prod_\perp$
$v_\parallel$	= particle speed, projected by $\prod_\parallel$
$(\underset{\sim}{x}, \underset{\sim}{\hat{p}}, t)$	= phase space variables
$\vec{x}$	= position variables
$Y_{\ell m}$	= spherical harmonic function
α	= expansion parameter in weak spatial gradients
$\hat{\beta}$	= unit vector in direction of mean magnetic field
$\vec{\beta}'$	= random magnetic field in units of the rms random field
γ	= Lorentz factor
Γ_1, Γ_2	= scalar, linear differential operators
$\widetilde{\Gamma}$	= rotated Γ
$\Gamma_{m,n}$	= matrix element in eigenvector expansion of Γ
$\gamma(0)$	= Lorentz factor at $\tau=0$
δ_{im}	= Kronecker delta function

$\Delta_1, \Delta_2, \Delta_n$	= decay constants in the uniformized asymptotic expansion
$\vec{\mathcal{E}}$	= total electric field
$\langle\vec{\mathcal{E}}\rangle$	= mean electric field
$\langle\vec{\mathcal{E}}'\rangle$	= random electric field
ϵ	= dimensionless inverse rigidity
ϵ_{ijk}	= Levi-Civita symbol of rank three
$\epsilon\eta$	= dimensionless expansion parameter which measures the strength of the particle-random field interaction
$\epsilon_{\mu\nu\lambda\rho}$	= Levi-Civita symbol of rank four
ζ	= radial particle displacement
$\mathcal{H}$	= Hamiltonian operator
η	= dimensionless rms value of random field
$\dot{\eta}_0, \dot{\eta}_1, \dot{\eta}_2$	= clock functions
$\mathcal{L}$	= generator of particle motion in mean field
$\mathcal{L}'$	= generator of particle motion in random field
$\tilde{\mathcal{L}}$	= rotated $\mathcal{L}$
κ_2	= linear, scalar, spatial differential operator
Λ	= finite-order matrix operator
$\Lambda_{\ell m}$	= coefficient in eigenvector expansion of Λ
ξ_0, ξ_1	= clock functions
$\dot{\xi}_0, \dot{\xi}_1$	= first derivatives of clock functions
$\prod_\perp$	= perpendicular projection operator with electric fields
$\prod_\parallel$	= parallel projection operator with electric fields

$\prod^1, \prod^2$ = linear combinations of $\prod_\perp$ and $\prod_\parallel$

$\mathscr{P}^1, \mathscr{P}^2$ = projection operators when $\vec{\mathcal{E}}\cdot\vec{B} = 0$

$\mathscr{P}$ = Cauchy principal value

$\mathscr{S}$ = finite-order matrix operator

$\tilde{\mathscr{S}}$ = rotated $\mathscr{S}$

$\dot{\tilde{\mathscr{S}}}$ = time derivative of $\hat{\mathscr{S}}$

σ = clock function

$\tau, \tau_0, \tau_1, \tau_2, \tau_n, \tau_{mn}$ = independent time scales in extended functions

$\tau_{\ell m}$ = decay time for order ℓm spherical harmonic in f

$\chi(\tau)$ = outer representation of $g(\tau)$

$\underset{\sim}{\Omega}$ = skew-symmetric rotation generator relative to $\hat{\beta}$

$\underset{\sim}{\Omega}'$ = skew-symmetric rotation generator relative to $\vec{\beta}'$

ω_B = gyrofrequency in mean field

$\omega_{\mu\nu}$ = field strength tensor

ω^* = dual of field strength tensor

$\omega_{\mathcal{E}}^2$ = inverse acceleration time in electric field

$\vec{\nabla}$ = gradient operator in position

$\vec{\phi}$ = cosmic ray flux

$\tilde{\Omega}$ = rotated Ω

SPECIAL PROBLEMS AND PUBLICATIONS

This report covers work carried out during the three-year period of January 12, 1968 to January 11, 1971. The main purpose of our program has been to implement Fermi's proposal that many important properties of cosmic rays should be understood in terms of the interactions of charged particles with the turbulent electric and magnetic fields contained in the large scale plasma that surrounds the Sun and penetrates the galactic disc. Our work on cosmic ray modulation is discussed thoroughly in this report. In addition, we have also been concerned with a number of special topics; in particular:

1 - Electrostatic effects in stellar winds.

2 - The heating of hydrogen clouds by cosmic rays and the possibility of giving a noncosmological explanation for the black-body background radiation.

3 - Statistical mechanics in relativistic form, including as a special case relativistic thermodynamics. This subject needs considerable clarification, especially when electric fields play a significant role. Even the simplest model (two-body relativistic potentials) presents substantial difficulties.

4 - A theoretical study of x-rays from the solar wind in the low-energy band.

5 - A feasibility analysis of a windowless x-ray counter whose main chamber is protected by a flow of gas through a fluid dynamic nozzle.

We include below a list of the publications that have been made during the past three years. The main body of the text following the list of publications discusses our work on the transport theory of cosmic rays.

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"Classical Relativistic Dynamics of Point Particles in External Potentials," G. Sandri and A. Klimas, Bull. Am. Phys. Soc. 13, 40 (1968).

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8. "Kinetic Equations for Cosmic Rays Valid for all Energies," G. Sandri and A. Klimas, Bull. Am. Phys. Soc. 15; 619 (1969).

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10. "Diffusion of Energetic Particles in a Turbulent Magnéto-plasma," A. Klimas and G. Sandri, Bull. Am. Phys. Soc. 15, 1426 (1970).

A.R.A.P. PUBLICATIONS

1. "Relativistic Kinetic Theory Suitable for Energetic Astrophysical Objects," A. Klimas and G. Sandri, August 1969.
2. "Power Law Spectra from Nonisothermal Sources," A. Klimas and G.Sandri, October 1969.
3. "Model for Charged Particle Motion in Random Magnetic Fields," A. Klimas and G. Sandri, October 1969.
4. "A Model for the Cosmic Ray Flux in a Strong Mean Magnetic Field with Scalar Damping," A. Klimas and G. Sandri, October 1969.
5. "A Model Equation for the Cosmic Ray Flux in a Strong Mean Magnetic Field with a Weak Random Part," A. Klimas and G. Sandri, October 1969.
6. "A Model Equation for the Cosmic Ray Flux in a Strong Mean Magnetic Field with a Weak Random Part: Linear Time Scales Extension, A. Klimas and G. Sandri, November 1969.
7. "Compatibility and Uniformity Conditions Applied to an Overextended Asymptotic Treatment of a Simple Problem," A. Klimas and G. Sandri, December 1969.
8. "Uniformization of the 2 x 2 Matrix Exponential with the Poincare-Lighthill Method," A. Klimas and G. Sandri, December 1969.
9. "Uniform Expansion of the 2 x 2 Matrix Exponential with Linear Time Scales," A. Klimas and G. Sandri, December 1969.
10. "Uniformization of Expansions for Linear Problems, A. Klimas and G. Sandri, February 1970.
11. "Relativistic Mechanics of Particles with a Two-body Interaction," A. Klimas and G. Sandri, February 1970.
12. "A Simple Example of Uniformization with Second-order Extension," A. Klimas and G. Sandri, March 1970.
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14. "A Cosmic Ray Transport Theory Valid over a Wide Range of Rigidity," A. Klimas and G. Sandri, May 1970.

15. "Cosmic Ray Transport Theory in Random Magnetic Fields," A. Klimas and G. Sandri, June 1970.

16. "Definitions and Properties of Some Differential Operators," A. Klimas and G. Sandri, September 1970.

17. "Kinetic Theory of Energetic Charged Particles in Random Magnetic Fields," A. Klimas and G. Sandri, October 1970.

CHAPTER I

INTRODUCTION - OUTLINE OF THE PROBLEM AND SUMMARY OF RESULTS

This report contains our development of the only cosmic ray transport theory available which is derived rigorously from first principles and is suitable for studies of the solar modulation problem. The only assumptions made in our development of this theory are: (1) we have neglected strong anisotropy in the cosmic ray beam, (2) we have neglected higher than two-point correlations in the random magnetic field, and (3) we have assumed the two-point correlation tensor to have an isotropic form.

The result of our analysis can be stated in the form of a convection-diffusion transport equation for the cosmic ray omnidirectional intensity, with the relationships between the transport coefficients and the mean and random magnetic fields given. We summarize this result here. The transport equation is

$$\frac{\partial I}{\partial T} + \vec{\nabla}\cdot(\vec{V}I) = \frac{1}{3}\,\vec{\nabla}\cdot\mathcal{D}\cdot\vec{\nabla}I \tag{1}$$

where I is the omnidirectional intensity which is a function of position, time, and rigidity. All lengths in Eq. (1) are measured in units of L, a macroscopic length which measures the size of the region in which the particles are contained; time is measured in units of L/v, where v is particle speed. The diffusion tensor has two components for diffusion perpendicular and parallel to the mean magnetic field; they are given by

$$D_{\perp} = \frac{-\omega_{\perp}}{(\epsilon+\omega_g)^2 + \omega_{\perp}^2} \tag{2}$$

and

$$D_{\parallel} = -\frac{1}{\omega_{\parallel}} \tag{3}$$

The convection velocity, $\vec{V}$, can be shown to be an average Alfvén drift velocity due to curvature and gradients in the mean field,

but modified due to the interaction with the random field. It is given by

$$\vec{V} = \vec{\nabla} \times (\hat{\beta} V) \tag{4}$$

where $\hat{\beta}$ is a unit vector in the direction of the mean field and

$$V = -\frac{1}{3}\left(\frac{\epsilon + \omega_g}{(\epsilon+\omega_g)^2+\omega_\perp^2}\right) \tag{5}$$

The quantities $\omega_\perp$, $\omega_\parallel$, and ω_g are related to the correlation function through the solution of the following dispersion relations

$$\omega_\parallel = -(\epsilon\eta)^2 \int_0^\infty d\lambda e^{-\omega_\parallel\lambda} K_\parallel(\epsilon,\lambda) \tag{6}$$

$$\omega_\perp = -(\epsilon\eta)^2 \int_0^\infty d\lambda e^{-\omega_\perp\lambda}\left[K_\perp(\epsilon,\lambda)\cos \omega_g\lambda - K_g(\epsilon,\lambda)\sin \omega_g\lambda\right] \tag{7}$$

and

$$\omega_g = (\epsilon\eta)^2 \int_0^\infty d\lambda e^{-\omega_\perp\lambda}\left[K_\perp(\epsilon,\lambda)\sin \omega_g\lambda + K_g(\epsilon,\lambda)\cos \omega_g\lambda\right] \tag{8}$$

The correlation function is written in the isotropic form

$$R_{ij}(\vec{r}) = \delta_{ij}A(r) + B(r)r_i r_j \tag{9}$$

We define

$$A_\ell(\epsilon,\lambda) \equiv \left(\frac{2\ell+1}{2}\right)\int_{-1}^{1} dzA(r)P_\ell(z) \tag{10}$$

and

$$B_\ell(\epsilon,\lambda) \equiv \left(\frac{2\ell+1}{2}\right)\int_{-1}^{1} dzB(r)P_\ell(z) \tag{11}$$

where P_ℓ is the ℓ'th-order Legendre polynomial and

$$r(\epsilon,\lambda,z) = \lambda\left[z^2+(1-z^2)\left(\frac{1 - \cos\,\epsilon\lambda}{\frac{1}{2}(\epsilon\lambda)^2}\right)\right]^{1/2} \tag{12}$$

We now introduce the following four combinations

$$J_1 = \frac{4\pi}{3}\left[A_0 - \frac{1}{5}\,A_2\right] \quad , \qquad J_2 = \frac{4\pi}{3}\left[\frac{2}{5}\,A_2 + A_0\right] \tag{13}$$

$$K_1 = \frac{4\pi}{15}\left[\frac{1}{21}\,B_4 - \frac{2}{7}\,B_2 + B_0\right] \tag{14}$$

$$K_2 = \frac{4\pi}{15}\left[-\frac{4}{21}\,B_4 + \frac{4}{2}\,B_2 + B_0\right] \tag{15}$$

The kernels in Eqs. (6) through (8) are given by

$$K_{\parallel}(\epsilon,\lambda) = 2J_1\cos\,\epsilon\lambda - 8\lambda^2K_1\left(\frac{\cos\,\epsilon\lambda-1}{\epsilon\lambda}\right)^2 \tag{16}$$

$$K_{\perp}(\epsilon,\lambda) = J_1+J_2\cos\,\epsilon\lambda-\lambda^2K_2\left[2\left(\frac{\sin\,\epsilon\lambda}{\epsilon\lambda}\right)\left(1-\frac{\sin\,\epsilon\lambda}{\epsilon\lambda}\right)-2\left(\frac{\cos\,\epsilon\lambda-1}{(\epsilon\lambda)^2}\right)-1\right] \tag{17}$$

and

$$K_g(\epsilon,\lambda) = J_2\sin\,\epsilon\lambda - \lambda^2K_2\left[2\left(\frac{\cos\,\epsilon\lambda-1}{\epsilon\lambda}\right)\left(1 - \frac{\sin\,\epsilon\lambda}{\epsilon\lambda}\right)\right] \tag{18}$$

These equations give a transport description of the cosmic ray particles in random magnetic fields. The complexity of these

equations is necessary to describe the low rigidities (less than 1 BV) in which modulation effects are observed. For very high rigidity, our results simplify considerably to $\omega_{\parallel} = \omega_{\perp} = -(\epsilon\eta)^2$ and $\omega_g = 0$.

There are two independent parameters in this theory. They are: ϵ, a measure of the inverse rigidity of the particles, and η, a measure of the strength of the fluctuations in the field. Our work is based on a truncated master equation for the cosmic ray distribution function which we derive from the Liouville equation using the technique of Kaufman.[1] This equation can be written schematically in the form

$$\frac{Df}{Dt} = (-\epsilon\eta)^2 C(\epsilon) f \tag{19}$$

where D/dt is a convective time derivative taken along the trajectory of the particles in the mean field and $C(\epsilon)$ is an integrodifferential operator which gives the effect of the random field on f . By deriving this equation from first principles, we have found that the right side of Eq. (19) represents the leading term of an expansion of the exact master equation in powers of $(\epsilon\eta)$. Higher-order terms in $(\epsilon\eta)$ bring in the effects of higher-order correlations in the random field. In the interplanetary field, ϵ=1 at approximately 1 BV . Since ϵ is inversely proportional to rigidity, we see that to discuss solar modulation effects we must develop a theory which is valid for $\epsilon \gtrsim 1$.

Previously developed theories of cosmic ray transport in random magnetic fields can be shown to be based on equations of the form of Eq. (19) which are, in some cases, identical to our truncated master equation and, in other cases, have been shown to be equivalent.[2,3,4,5] However, in all of these theories, when modulation energies are discussed, the approximate effects of $C(\epsilon)$ are determined by expanding C in powers of $1/\epsilon$ and retaining the leading term. These expansions are in clear

violation of the condition $(\epsilon\eta)^2 \ll 1$ unless $\eta \to 0$. In the transport theory which we have presented in the preceding paragraphs, no expansion in ϵ has been made. Thus, the theory is valid in the range $\epsilon \sim 1$.

In the remainder of this report, we describe the derivation of our transport theory. In Chapter II, we formulate the problem of the transport of cosmic rays in full generality from first principles. We keep explicitly only magnetic fields, though electric fields could be introduced in most of this chapter without great difficulty. The problem is set up in terms of the Liouville equation which is then ensemble-averaged over the distribution of magnetic fields. A master equation which constitutes the foundation of our work is deduced which allows for a closed formulation of the transport problem. Two special cases are then discussed. In the ultrahigh energy regime, contact is established with conventional theory. We then resum the set of contributions that corresponds to only two-point correlations with straight line trajectories. This resummed equation has an H-theorem and allows us to extrapolate from ultrahigh to high energies.

In Chapter III, we deduce the equations for the omnidirectional intensity and flux as the first two spherical harmonic components of the velocity distribution function discussed in the previous chapter. These equations are the basis for the transport theory in the modulation energy range. With the techniques of the next two chapters, we have deduced the new convection-diffusion equation given here.

In Chapter IV, we develop the asymptotic expansion for systems of linear equations. The techniques developed in this chapter apply to the master equation of Chapter II, as well as to the equations of Chapter III. We show how to extract the secularities and how to obtain regular expansions.

We find that for short-range two-body correlation functions, when $\epsilon \sim 1$, the flux equation is an integrodifferential equation with a long-range kernel. Kernels with short range have been treated previously and the available uniformization techniques are adequate. The special problems presented by long-range kernels are discussed and solved in Chapter V.

In Chapter VI, we introduce electric as well as magnetic fields. In particular, we obtain complete solutions for the problem of charged particles moving in uniform fields. Two Lorentz-invariant, gauge-invariant constants of the motion are obtained which turn out to be adiabatic invariants. We, thus, have laid the groundwork for an expansion in small electric fields (such as those in Alfvén waves).

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CHAPTER II

GENERAL FORMULATION OF THE PROBLEM - ULTRAHIGH AND HIGH-ENERGY APPROXIMATIONS

Following Fermi's[1] original paper in which he suggested that the cosmic ray flux is of galactic origin but still isotropic due to the presence of a disordered galactic magnetic field, a great deal of work has been done to refine our understanding of the transport of energetic particles in random electromagnetic fields. In the earliest of this work, transport equations were assumed which usually had one feature in common: the connection between the relevant transport coefficients and the state of the random field was only loosely modeled.[2] Attempts to strengthen the connection between the transport coefficients and the random field have been made, quite naturally in studies of the kinetic theory which might underlie the cosmic ray transport theory. In some cases,[3] the Boltzmann equation has been adopted, but then it has become necessary to devise an effective cross section for the particle-random field interaction. Even so, this approach has led to transport theories in which all the transport coefficients could, at least, be related to the assumed cross section. A major advance was made when several investigators introduced the Fokker-Planck equation as a possible kinetic theory.[4] In the transport theories which resulted, the transport coefficients could be directly related to measurable properties of the random field.

Recently, attempts have been made to justify the assumption of the Fokker-Planck equation as the relevant kinetic theory for energetic particles in random electromagnetic fields. These attempts have been based on the more fundamental Liouville equation or its equivalent. Hall and Sturrock,[5] using the quasilinear approximation, have constructed a kinetic theory from the Liouville equation which they have demonstrated to be equivalent to the earlier Fokker-Planck theories. They use a test particle analysis in which the effects of the moving charged particles on the electromagnetic field are neglected. Kulsrud and Pearce[6]

have developed a more general, self-consistent theory in which the field can be affected by the energetic particles. In both of these theories, under suitable conditions, the authors have shown that they are able to reproduce the results of Jokipii[4] which are based on the Fokker-Planck equation. The essential conditions which are necessary for these comparisons are: (1) a stationary random magnetic field with no electric field and (2) low rigidity particles so that the radius of gyration of the particles is negligibly small compared to the correlation length associated with the random field.

The work which we present in this report falls into the class of papers in which the appropriate cosmic ray kinetic theory is derived from the Liouville equation. Our work does not embody all of the generality of the previous papers in this class; we limit ourselves to a test particle analysis of particles moving in a stationary magnetic field with no electric field. Early in our work, we reproduce a special case of the kinetic theory of Hall and Sturrock; however, our approach is considerably different. We do not use the quasi-linear approximation at the outset; and, as a result, we are able to determine more exactly the conditions under which this kinetic theory is a valid approximation. We find these conditions in contradiction to the assumption of low rigidity particles which is made by Jokipii, Hall and Sturrock, and Kulsrud and Pearce when they attempt to evaluate the Fokker-Planck coefficients or their equivalent. This is not to say that the Fokker-Planck or the equivalent kinetic theories do not apply; the problem lies in further approximations which have been made in the evaluations of the Fokker-Planck coefficients.

In our approach, following the technique of Kaufman,[7] we construct from the Liouville equation the exact master equation for the ensemble-averaged, one-particle distribution function. We then expand the master equation in powers of the magnitude of the random field. Stopping at second order, we obtain the

kinetic theory of Hall and Sturrock with no electric fields, which is equivalent to the Fokker-Planck theory as well. The conditions under which the relevant expansion parameter is small can be stated as follows. Let us measure the particle radius of gyration in the mean field in units of the two-point correlation length in the random field. When the radius of gyration is of the order of the correlation length, then $r_g \sim 1$. If η is the ratio of the root-mean-square value of the random field to the magnitude of the mean field, then the Fokker-Planck theory is a valid approximation so long as $r_g \gg \eta$. The previous attempts of Jokipii, Hall and Sturrock, and Kulsrud and Pearce to evaluate the Fokker-Planck coefficients by obtaining the leading-order term in an expansion of the coefficients in r_g about $r_g=0$ is clearly in violation of this condition.

In Section 1, immediately following this Introduction, we derive the master equation. In Section 2, we expand the master equation to second order in $\epsilon\eta$. Also, in Section 2, we expand our "interaction operator," the equivalent of the Fokker-Planck coefficients, up to n'th order in ϵ and then resum. We use the time-scale extension technique[8,9] to ensure the compatibility and uniformity of our expansions; a simple perturbation expansion fails for large times. In Section 3, we note that an H-theorem can be calculated from our kinetic theory; an arbitrary initial distribution function always relaxes to a final, isotropic state. In addition, we calculate the relaxation time necessary for the final state to be reached. From these calculations, we find an additional constraint on our theory. Distribution functions which are very anisotropic cannot be described by it.

1. DERIVATION OF THE MASTER EQUATION

We ignore the effects of interparticle collisions and, instead, concentrate on the behavior of charged particles in a magnetic field characterized by a field ensemble mean part, $\langle \underset{\sim}{B} \rangle$, and a random part, $\underset{\sim}{B}'$. Then the Liouville equation for the N-body distribution function reduces to

$$\frac{\partial F^{(1)}}{\partial \tau} + KF^{(1)} + \epsilon(\mathcal{L} + \eta\mathcal{L}')F^{(1)} = 0 \tag{1.1}$$

where $F^{(1)}(\underset{\sim}{x},\hat{\underset{\sim}{p}},\tau)$ is the one-body distribution function which depends on position, $\underset{\sim}{x}$, direction on the unit-momentum sphere, $\hat{\underset{\sim}{p}}$, and time τ. The differential operators which appear in Eq. (1.1) have all been written in dimensionless forms in the following manner. A length, λ, has been introduced which is a measure of the two-point correlation length in $\underset{\sim}{B}'$. Time is measured in units of the traversal time, λ/v, where v is the particle speed. Thus

$$\tau = t(v/\lambda) \tag{1.2}$$

Spatial gradients in f are measured in units of λ; therefore

$$K = \hat{\underset{\sim}{p}} \cdot \frac{\partial}{\partial(\underset{\sim}{x}/\lambda)} \tag{1.3}$$

The Lorentz operators $\mathcal{L}$ and $\mathcal{L}'$ are written as

$$\mathcal{L} = -\hat{\underset{\sim}{p}} \cdot \underset{\sim}{\Omega} \cdot \frac{\partial}{\partial \hat{\underset{\sim}{p}}} \tag{1.4}$$

and

$$\mathcal{L}' = -\hat{\underset{\sim}{p}} \cdot \underset{\sim}{\Omega}' \cdot \frac{\partial}{\partial \hat{\underset{\sim}{p}}} \tag{1.5}$$

where $\underset{\sim}{\Omega}$ and $\underset{\sim}{\Omega}'$ are the skew-symmetric tensors defined by

$$\Omega_{ij} = \epsilon_{ijk}\beta_k = \epsilon_{ijk}\frac{\langle B_k\rangle}{\langle B\rangle} \tag{1.6}$$

and

$$\Omega'_{ij} = \epsilon_{ijk}\beta'_k = \epsilon_{ijk}\frac{B'_k}{\sqrt{\langle \underset{\sim}{B}'\cdot\underset{\sim}{B}'\rangle}} \tag{1.7}$$

The carat denotes the unit vector, $\hat{p} = \vec{p}/p$, and, in addition, we use the notation $\partial/\partial\hat{p}_i = p(\delta_{ij} - \hat{p}_i\hat{p}_j)(\partial/\partial p_j)$. Both $\mathcal{L}$ and $\mathcal{L}'$ do not operate on functions of the magnitude of the momentum. Dynamical variables in momentum reduce to the two directions on the unit-momentum sphere.

As a result of this dimensional analysis, two parameters emerge. The first of these

$$\eta = \sqrt{\langle \underset{\sim}{B}'\cdot\underset{\sim}{B}'\rangle}/\langle B\rangle \tag{1.8}$$

is a measure of the relative strength of the random part of the field. The second parameter is

$$\epsilon = \frac{\lambda\langle B\rangle}{P} \tag{1.9}$$

where P is the particle rigidity. ϵ can be thought of as the ratio of the correlation length to the gyro-radius of the particle in the mean field.

We look for solutions to Eq. (1.1) of the form

$$F^{(1)} = f + g \tag{1.10}$$

where f is the field ensemble average of $F^{(1)}$ and g represents the fluctuations in $F^{(1)}$. To determine the relevant kinetic theory for f , we further assume that f is independent of position. In a later paper, we will allow weak gradients to determine the transport properties of the particles in the random magnetic field.

Following the technique of Kaufman (1968) we have determined from Eq. (1.1) that f obeys the following master equation

$$\frac{\partial f}{\partial \tau} + \epsilon \mathcal{L} f = (\epsilon\eta)^2 (1 - \epsilon\eta \langle \mathcal{L}' G \rangle)^{-1} \langle \mathcal{L}' G \mathcal{L}' \rangle f \tag{1.11}$$

where

$$G^{-1} = \frac{\partial}{\partial \tau} + K + \epsilon \mathcal{L} + \epsilon\eta \mathcal{L}' \tag{1.12}$$

Thus, G is the Green's function operator with the total Hamiltonian

$$\mathcal{H} = K + \epsilon \mathcal{L} + \epsilon\eta \mathcal{L}' \tag{1.13}$$

for the generator of the particle motion. Before going on, we note that Eq. (1.11) is considerably more general than implied here. Any force field for which a Hamiltonian for the particle motion can be constructed could be included in the formalism which leads to Eq. (1.11). In particular, in subsequent work, we plan to include electric fields with random parts.

2. EXPANSION IN $\epsilon\eta$ WITH LINEAR TIME SCALES EXTENSION

Given the master equation for f from the previous section, we notice that it contains two independent parameters ϵ and $\epsilon\eta$. In this section, we construct an asymptotic expansion of f to second order in $\epsilon\eta$. We employ the linear time scale extension technique to ensure the uniform validity of the expansion for large time.

A power series expansion of the master equation in $\epsilon\eta$ is given by

$$\frac{\partial f}{\partial \tau} + \epsilon \mathcal{L} f = (\epsilon\eta)^2 \langle \mathcal{L}' G_0 \mathcal{L}' \rangle f + O(\epsilon\eta)^3 \tag{2.1}$$

where G_0 is the Green's function integral operator which inverts

$$G_0^{-1} = \frac{\partial}{\partial \tau} + K + \epsilon \mathcal{L} \tag{2.2}$$

The generator of the particle motion is $\mathcal{H} = K + \epsilon\mathcal{L}$, the Hamiltonian operator in the mean field. Notice that the second-order interaction operator $\langle \mathcal{L}' G_0 \mathcal{L}' \rangle$ involves only the two-point correlation function in the random field. Thus, an expansion of f to second order in $\epsilon\eta$ neglects the effects of higher-order field correlations.

A generalization of Eq. (2.1) in which electric fields have been included has been derived by Hall and Sturrock.[5] They use the quasi-linear approach to go directly from the Liouville equation to their version of Eq. (2.1), bypassing the construction of the master equation. They have also pointed out the equivalence of the Fokker-Planck equation and Eq. (2.1) to second order in $\epsilon\eta$. Indeed, they find that their theory for stationary magnetic fields and in the limit of vanishing rigidity reduces to the low rigidity work of Jokipii[4] which is based on the Fokker-Planck equation. Kulsrud and Pearce find that under the same conditions, which they adopt for most of their paper, their Fokker-Planck theory of cosmic ray propagation reduces also to the low rigidity results of Jokipii.

We find ourselves in disagreement with the low rigidity results of these previous theories. By deriving Eq. (2.1) from the exact master equation, we find the specific condition for its validity as an approximation, $(\epsilon\eta)^2 \ll 1$ The low rigidity results of Hall and Sturrock, Jokipii, and Kulsrud and Pearce are computed asymptotically in the limit of vanishing $1/\epsilon$, where Eq. (2.1) itself is not correct.

Hall and Sturrock and Kulsrud and Pearce have not made specific comparisons of their theories with the high-rigidity results of Jokipii which are calculated asymptotically in the limit of vanishing ϵ . On comparing our results with those of Jokipii's, we find that the first term of our ϵ-expansion gives his high rigidity results. Thus, our theory is in agreement with Jokipii's asymptotically in the limit of infinite rigidity; our theory contains significant corrections to Jokipii's when $\epsilon \sim 1$ (when the radius of gyration and the correlation length in the random field are of the same order of magnitude) and our theory is in definite disagreement with the expansions in $1/\epsilon$.

We have attempted a perturbation expansion of f in powers of $\epsilon\eta$ and ϵ but have found it nonuniform in time. The formal ordering in the expansion fails for large time because of secular growth of the higher-order terms. To obtain a uniform expansion of f , we employ the time scale extension technique[8,9] to remove this secular behavior from the perturbation expansion. In the following, the nonuniform behavior of the perturbation terms will be clearly exhibited.

We introduce an extended distribution function, $\underline{f}$, which is a function of many independent time scales $(\tau_0, \tau_1, \tau_2, \ldots)$. $\underline{f}$ is considered an extension of f if a restricted trajectory $(\tau_0(\tau), \tau_1(\tau), \tau_2(\tau), \ldots)$ can be found in the multidimensional time-scale space along which $\underline{f}(\tau_0, \tau_1, \tau_2, \ldots)$ reduces to $f(\tau)$ We further expand $\underline{f}$ in powers of $\epsilon\eta$

$$\underline{f} = \underline{f}_0 + \epsilon\eta \underline{f}_1 + (\epsilon\eta)^2 \underline{f}_2 + \cdots \tag{2.3}$$

and use the freedom we have in choosing a restricted trajectory in the time-scale space to ensure the expansion, as determined from Eq. (2.1), is uniformly valid in time.

We have found that a linear trajectory is suitable for removing nonuniform terms from the expansion of $\underline{f}$ in Eq. (2.3). The trajectory is given by

$$\tau_n = (\epsilon\eta)^n \tau \qquad (2.4)$$

and along this trajectory

$$\frac{\partial}{\partial\tau} = \frac{\partial}{\partial\tau_0} + (\epsilon\eta)\frac{\partial}{\partial\tau_1} + (\epsilon\eta)^2\frac{\partial}{\partial\tau_2} + \cdots \qquad (2.5)$$

The extension of Eq. (2.1) is obtained by substituting Eqs. (2.3) and (2.5) into Eq. (2.1); we do not extend the interaction operator. By combining the coefficients of each power of $(\epsilon\eta)$, we find to second order

$$\frac{\partial \underline{f}_0}{\partial\tau_0} + \epsilon\mathcal{L}\underline{f}_0 = 0 \qquad (2.6)$$

$$\frac{\partial \underline{f}_1}{\partial\tau_0} + \frac{\partial \underline{f}_0}{\partial\tau_1} + \epsilon\mathcal{L}\underline{f}_1 = 0 \qquad (2.7)$$

and

$$\frac{\partial \underline{f}_2}{\partial\tau_0} + \frac{\partial \underline{f}_1}{\partial\tau_1} + \frac{\partial \underline{f}_0}{\partial\tau_2} + \epsilon\mathcal{L}\underline{f}_2 = \langle\mathcal{L}' G_0 \mathcal{L}'\rangle \underline{f}_0 \qquad (2.8)$$

From Eq. (2.6) we find

$$\underline{f}_0(\vec{\tau}_0) = e^{-\epsilon\mathcal{L}\tau_0}\,\underline{f}_0(\vec{\tau}_1) \qquad (2.9)$$

We have used the following notation: $\underline{f}_0(\vec{\tau}_0)$ gives the dependence of $\underline{f}_0$ on τ_0 and all higher-order time scales to be introduced while $\underline{f}_0(\vec{\tau}_1)$ gives the dependence of $\underline{f}_0$ on τ_1 and all the higher-order time scales but with $\tau_0 = 0$. By substituting Eq. (2.9) into Eq. (2.7) and integrating over τ_0, we find

$$\underline{f}_1(\vec{\tau}_0) = -\tau_0 \frac{\partial \underline{f}_0(\vec{\tau}_0)}{\partial\tau_1} \qquad (2.10)$$

To remove this secular behavior on the τ_0 scale from $\underline{f}_1$ we set

$$\underline{f}_1(\vec{\tau}_0) = 0 \tag{2.11}$$

and

$$\frac{\partial \underline{f}_0(\vec{\tau}_0)}{\partial \tau_1} = 0 \tag{2.12}$$

In this case, Eq. (2.8) reduces to

$$\frac{\partial \underline{f}_2}{\partial \tau_0} + \epsilon \mathcal{L} \underline{f}_2 = - \frac{\partial \underline{f}_0}{\partial \tau_2} + \langle \mathcal{L}' G_0 \mathcal{L} \rangle \underline{f}_0 \tag{2.13}$$

We have been unable to evaluate the secular behavior of the perturbation term in Eq. (2.13) directly. The difficulty is related to the helical trajectory generated by H. On the other hand, a further expansion of G_0 in powers of ϵ is possible. We find

$$G_0[1 + \epsilon G_{00}\mathcal{L}]^{-1} G_{00} = \sum_{n=0}^{\infty} (-\epsilon)^n (G_{00}\mathcal{L})^n G_{00} \tag{2.14}$$

where the generator of the particle motion in G_{00} is simply K; the particle moves along a straight line at constant momentum. This simplification in the particle trajectory is actually sufficient to overcome the added complexity introduced by the infinite summation in Eq. (2.14). Upon expanding $\underline{f}_0$ and $\underline{f}_2$ in powers of ϵ, we find we are able to evaluate the secular behavior of each term in this expansion of the interaction operator. We find that a nested time-scale extension in ϵ is necessary to ensure the uniformity of the expansions of $\underline{f}_0$ and $\underline{f}_2$

Substituting Eq. (2.14) into Eq. (2.13), we find

$$\frac{\partial \underline{f}_2}{\partial \tau_0} + \epsilon \mathcal{L} \underline{f}_2 = - \frac{\partial \underline{f}_0}{\partial \tau_2} + \sum_{n=0}^{\infty} (-\epsilon)^n \langle \mathcal{L}' (G_{00}\mathcal{L})^n G_{00} \mathcal{L}' \rangle \underline{f}_0 \tag{2.15}$$

where

$$\langle \mathcal{L}'(G_{00}\mathcal{L})^n G_{00}\mathcal{L}\rangle f_0 = \langle \mathcal{L}'(\vec{x},\hat{p}) \int_0^{\tau_0} ds \mathcal{K}_n(s)\mathcal{L}'(\vec{x}-\hat{p}s,\hat{p})\rangle \underline{f}_0(\hat{p},\vec{s}) \quad (2.16)$$

with

$$\mathcal{K}_n(s) = \int_0^{s} d\lambda \mathcal{K}_{n-1}(\lambda)\tilde{\mathcal{L}}(\lambda) \quad (2.17)$$

for $n > 0$ and with $\mathcal{K}_0(s) = 1$. The quantity $\tilde{\mathcal{L}}$ is defined by

$$\tilde{\mathcal{L}}(\tau) \equiv e^{-K\tau}\mathcal{L}e^{K\tau} \quad (2.18)$$

which reduces to

$$\tilde{\mathcal{L}}(\tau) = \mathcal{L} - \tau\kappa_2 \quad (2.19)$$

where

$$\kappa_2 = \hat{p}\cdot\Omega\cdot\vec{\nabla} = \hat{p}\cdot\Omega\cdot\frac{\partial}{\partial\vec{x}/\lambda} \quad (2.20)$$

We now expand $\underline{f}_0$ and $\underline{f}_2$ in powers of ϵ so that

$$\underline{f}_0 = \underline{f}_{00} + \epsilon\underline{f}_{01} + \epsilon^2\underline{f}_{02} + \dots \quad (2.21)$$

and

$$\underline{f}_2 = \underline{f}_{20} + \epsilon\underline{f}_{21} + \epsilon^2\underline{f}_{22} + \dots \quad (2.22)$$

We further extend these functions and choose the trajectory

$$\tau_{mn} = (\epsilon\eta)^m\epsilon^n\tau \quad (2.23)$$

so that, in particular,

$$\frac{\partial}{\partial\tau_0} = \frac{\partial}{\partial\tau_{00}} + \epsilon\frac{\partial}{\partial\tau_{01}} + \epsilon^2\frac{\partial}{\partial\tau_{02}} + \cdots \tag{2.24}$$

and

$$\frac{\partial}{\partial\tau_2} = \frac{\partial}{\partial\tau_{20}} + \epsilon\frac{\partial}{\partial\tau_{21}} + \epsilon^2\frac{\partial}{\partial\tau_{22}} + \cdots \tag{2.25}$$

From Eq. (2.6), we find

$$\frac{\partial \underline{f}_{00}}{\partial\tau_{00}} = 0 \tag{2.26}$$

and then

$$\frac{\partial \underline{f}_{01}}{\partial\tau_{00}} + \frac{\partial \underline{f}_{00}}{\partial\tau_{01}} + \mathcal{L}\underline{f}_{00} = 0 \tag{2.27}$$

Since $\underline{f}_{00}$ does not depend on τ_{00} ,

$$\underline{f}_{01} = -\tau_{00}\left[\frac{\partial \underline{f}_{00}}{\partial\tau_{01}} + \mathcal{L}\underline{f}_{00}\right] \tag{2.28}$$

Notice that we are restricting our choice of initial conditions to

$$\underline{f}_{0i}(\tau_{00}=0) = 0 \tag{2.29}$$

for all $i > 0$. To remove the secular growth in $\underline{f}_{01}$, we must let

$$\frac{\partial \underline{f}_{00}}{\partial\tau_{01}} + \mathcal{L}\underline{f}_{00} = 0 \tag{2.30}$$

and then

$$\underline{f}_{01} = 0 \tag{2.31}$$

Again, from Eq. (2.6)

$$\frac{\partial \underline{f}_{02}}{\partial\tau_{00}} + \frac{\partial \underline{f}_{00}}{\partial\tau_{02}} = 0 \tag{2.32}$$

Therefore

$$\underline{f}_{02} = -\tau_{00} \frac{\partial \underline{f}_{00}}{\partial \tau_{02}} \tag{2.33}$$

We let

$$\frac{\partial \underline{f}_{00}}{\partial \tau_{02}} = 0 \qquad \text{and} \qquad \underline{f}_{02} = 0 \tag{2.34}$$

This process can be repeated to obtain

$$\underline{f}_{0i} = 0 \qquad i = 1,2,3\ldots \tag{2.35}$$

and

$$\frac{\partial \underline{f}_{00}}{\partial \tau_{0i}} = 0 \qquad i = 2,3,\ldots \tag{2.36}$$

From Eqs. (2.15), (2.16) and (2.17), we find to zeroth order in ϵ

$$\frac{\partial \underline{f}_{20}}{\partial \tau_{00}} = -\frac{\partial \underline{f}_{00}}{\partial \tau_{20}} + \left\langle \mathcal{L}'(\vec{x},\hat{p}) \int_0^{\tau_{00}} ds \mathcal{K}_0(s) \mathcal{L}'(\vec{x}-\hat{p}s,\hat{p}) \right\rangle \underline{f}_{00}(\hat{p}) \tag{2.37}$$

where, from Eq. (2.26), $\underline{f}_{00}$ does not depend on the variable of integration, s. In Appendix A we show that if

$$\vec{r} = \vec{x}' - \vec{x} = -\hat{p}s \tag{2.38}$$

and if we choose the isotropic form for the two-point correlation function for the random field[10]

$$R_{ij}(\vec{r}) = A(r)\delta_{ij} + B(r)\, r_i r_j \tag{2.39}$$

where A and B are arbitrary even functions of r, then

$$\langle \mathcal{L}'(\vec{x},\hat{p}) \mathcal{L}'(\vec{x}',\hat{p}) \rangle = A(r)\hat{\nabla}^2 \tag{2.40}$$

where

$$\hat{\nabla}^2 = \frac{\partial}{\partial \hat{p}} \cdot \frac{\partial}{\partial \hat{p}} \tag{2.41}$$

is the angular part of the Laplacian operator in spherical momentum coordinates. Thus, we find

$$\frac{\partial \underline{f}_{20}}{\partial \tau_{00}} = - \frac{\partial \underline{f}_{00}}{\partial \tau_{20}} + \int_0^{\tau_{00}} ds A(s) \hat{\nabla}^2 \underline{f}_{00} \tag{2.42}$$

We assume that $A(s)$ is sharply cut off at $s \simeq 1$ so that

$$\int_0^{\tau_{00}} ds s^n A(s) \tag{2.43}$$

exists and is finite for all n as $\tau_{00} \to \infty$. On the other hand, if

$$\int_0^{\tau_{00}} ds A(s)$$

reaches a constant finite value for $\tau_{00} \gtrsim 1$, then integration of Eq. (2.42) over τ_{00} to obtain $\underline{f}_{20}$ will lead to growth of the perturbation term directly proportional to τ_{00} for large τ_{00} . The time scale extension technique has been introduced by us to remove this secular behavior of the perturbation terms. We set

$$\frac{\partial \underline{f}_{00}}{\partial \tau_{20}} = \int_0^{\infty} ds A(s) \hat{\nabla}^2 \underline{f}_{00} \tag{2.44}$$

and

$$\underline{f}_{20}(\vec{\tau}_{00}) = \underline{f}_{20}(\vec{\tau}_{01}) - \int_0^{\tau_{00}} ds \int_s^{\infty} ds' A(s') \hat{\nabla}^2 \underline{f}_{00} \tag{2.45}$$

which leads to a finite $\underline{f}_{20}(\tau_{00})$ (and, therefore, a uniform expansion of $\underline{f}_2$ to this order in ϵ and further gives us the leading kinetic equation for $\underline{f}_0$ in ϵ . The nonzero $\underline{f}_{20}(\vec{\tau}_{01})$ must be reatined to allow a uniform expansion in higher orders of

To first order in ϵ , we find from Eq. (2.15)

$$\frac{\partial \underline{f}_{20}}{\partial \tau_{01}} + \mathcal{L}\underline{f}_{20} + \frac{\partial \underline{f}_{21}}{\partial \tau_{00}} = - \frac{\partial \underline{f}_{00}}{\partial \tau_{21}} + \\ - \langle \mathcal{L}'(\vec{x},\hat{p}) \int_0^{\tau_{00}} ds \mathcal{K}_1(s) \mathcal{L}'(\vec{x}-\hat{p}s,\hat{p}) \rangle \underline{f}_{00}(\hat{p}) \quad (2.46)$$

By direct computation, we find that

$$[\hat{\nabla}^2_p, \mathcal{L}] = 0 \quad (2.47)$$

Therefore, from Eqs. (2.45) and (2.30),

$$\frac{\partial \underline{f}_{20}(\vec{\tau}_{00})}{\partial \tau_{01}} + \mathcal{L}\underline{f}_{20}(\vec{\tau}_{00}) = \frac{\partial \underline{f}_{20}(\vec{\tau}_{01})}{\partial \tau_{01}} + \mathcal{L}\underline{f}_{20}(\vec{\tau}_{01}) \quad (2.48)$$

The first-order kernel is

$$\mathcal{K}_1(s) = s\mathcal{L} - \frac{1}{2} s^2 \kappa_2 \quad (2.49)$$

To evaluate the effect of κ_2 in the interaction term, we write

$$\langle \mathcal{L}'(\vec{x},\hat{p}) \kappa_2 \mathcal{L}'(x',\hat{p}) \rangle = - \frac{\partial}{\partial \hat{p}_i} \hat{p}_j \langle \Omega'_{ij}(\vec{x}) \kappa_2 \Omega'_{\ell m}(\vec{x}') \rangle \hat{p}_\ell \frac{\partial}{\partial \hat{p}_m} \quad (2.50)$$

Now κ_2 acting on $\Omega'(\vec{x}-\hat{p}s)$ can be written as $\frac{1}{s}\mathcal{L}$ acting on $\Omega'(\vec{x}-\hat{p}s^2)$. Therefore

$$\langle \mathcal{L}'(\vec{x},\hat{p}) \kappa_2 \mathcal{L}'(\vec{x}',\hat{p}) \rangle = - \frac{\partial}{\partial \hat{p}_i} \hat{p}_j \Big[\big| \mathcal{L} \langle \Omega'_{ij}(\vec{x}) \Omega'_{\ell m}(\vec{x}') \rangle \big| \Big] \hat{p}_\ell \frac{\partial}{\partial \hat{p}_m} \quad (2.51)$$

where the symbol $\Big[\big| \;\; \big| \Big]$ signifies that $\mathcal{L}$ acts only on the quantities within the bracket and not on everything to the right. The expectation value can be evaluated and on carrying out the rest of the operations indicated in Eq. (2.51) we find

$$\langle \mathcal{L}'(\vec{x},\hat{p})\kappa_2 \mathcal{L}'(\vec{x}',p)\rangle = 0 \tag{2.52}$$

Thus

$$\frac{\partial \underline{f}_{20}(\vec{\tau}_{01})}{\partial \tau_{01}} + \mathcal{L}\underline{f}_{20}(\vec{\tau}_{01}) + \frac{\partial \underline{f}_{21}}{\partial \tau_{00}} = - \frac{\partial \underline{f}_{00}}{\partial \tau_{21}} +$$

$$- \langle \mathcal{L}'(\vec{x},\hat{p}) \int_0^{\tau_{00}} ds s \mathcal{L}\mathcal{L}'(\vec{x}-\hat{p}s,\hat{p})\rangle \underline{f}_{00}(\hat{p}) \tag{2.53}$$

and

$$\underline{f}_{21}(\vec{\tau}_{00}) = \underline{f}_{21}(\vec{\tau}_{01}) - \tau_{00}\left(\frac{\partial \underline{f}_{20}(\vec{\tau}_{01})}{\partial \tau_{01}} + \mathcal{L}\underline{f}_{20}(\vec{\tau}_{01})\right) +$$

$$- \tau_{00}\frac{\partial \underline{f}_{00}}{\partial \tau_{21}} - \int_0^{\tau_{00}} ds \int_0^{s} ds' \langle \mathcal{L}'(\vec{x},\hat{p}) s' \mathcal{L}\mathcal{L}'(\vec{x}-\hat{p}s',\hat{p})\rangle \underline{f}_{00} \tag{2.54}$$

The procedure at this point is as follows. The perturbation term can be written in the general form

$$Q_1(\tau_{00})\underline{f}_{00} - \tau_{00}(\gamma_1 + \Delta_1)\underline{f}_{00} \tag{2.55}$$

where $Q_1(\tau_{00})$ is a bounded function of τ_{00} and γ_1 and Δ_1 are τ_{00} independent. They have been differentiated by the condition

$$\begin{aligned} [\hat{\nabla}^2, \Delta_1] &= 0 \\ [\hat{\nabla}^2, \gamma_1] &\neq 0 \end{aligned} \tag{2.56}$$

We set

$$\underline{f}_{21}(\vec{\tau}_{00}) = \underline{f}_{21}(\vec{\tau}_{01}) - Q_1(\tau_{00})\underline{f}_{00} \tag{2.57}$$

$$\frac{\partial \underline{f}_{20}(\vec{\tau}_{01})}{\partial \tau_{01}} + \mathcal{L}\underline{f}_{20}(\vec{\tau}_{01}) = \gamma_1 \underline{f}_{00} \tag{2.58}$$

and

$$\frac{\partial \underline{f}_{00}}{\partial \tau_{21}} = \Delta_1 \underline{f}_{00} \tag{2.59}$$

This separation of terms is dictated by the condition that $\underline{f}_{21}(\vec{\tau}_{00})$ be bounded and by the compatibility condition[9]

$$\frac{\partial^2 \underline{f}_{00}}{\partial \tau_{00} \partial \tau_{21}} - \frac{\partial^2 \underline{f}_{00}}{\partial \tau_{21} \partial \tau_{00}} = [\hat{\nabla}^2, \Delta_1] \underline{f}_{00} = 0 \tag{2.60}$$

The secular part of the perturbation term which commutes with $\hat{\nabla}^2$ determines the kinetic behavior of $\underline{f}_{00}$ on the τ_{21} scale, while the secular but noncommuting part determines $\underline{f}_{20}(\vec{\tau}_{01})$. Since we are primarily interested in the kinetic behavior of $\underline{f}_{00}$, we shall not specify the Q's and γ's in this and higher orders in ϵ (γ_1 is actually zero but higher-order γ's are not) but we will give the Δ_n. We find

$$\frac{\partial \underline{f}_{00}}{\partial \tau_{21}} = - \langle \mathcal{L}'(\vec{x}, \hat{p}) \int_0^{\infty} ds s \mathcal{L}\mathcal{L}'(\vec{x}, \hat{p}) \rangle \underline{f}_{00} \tag{2.61}$$

The fact that this Δ_1 commutes with $\hat{\nabla}^2$ is not made clear here, but this point will be covered later in this calculation.

By using similar reasoning to higher orders in ϵ, we have found

$$\frac{\partial \underline{f}_{00}}{\partial \tau_{2n}} = \Delta_n \underline{f}_{00} \tag{2.62}$$

where

$$\Delta_n = \frac{(-1)^n}{n!} \langle \mathcal{L}'(\vec{x}, \hat{p}) \int_0^{\tau_{00}} ds s^n \mathcal{L}^n \mathcal{L}'(\vec{x} - \hat{p}s, \hat{p}) \rangle \tag{2.63}$$

In Appendix B we show that

$$\mathcal{L}'\mathcal{L}^n = \sum_{k=0}^{n} \frac{(-1)^k n!}{(n-k)!k!} \mathcal{L}^{n-k}[\mathcal{L},\mathcal{L}']_k \tag{2.64}$$

where we have adapted the notation

$$[\mathcal{L},\mathcal{L}'] = [\mathcal{L},\mathcal{L}']_1 \quad ; \qquad [\mathcal{L},[\mathcal{L},\mathcal{L}']] = [\mathcal{L},\mathcal{L}']_2 \tag{2.65}$$

so that $[\mathcal{L},\mathcal{L}']_k$ stands for the k'th commutation of $\mathcal{L}$ with $\mathcal{L}'$.

In Appendix C we show that

$$[\mathcal{L},\mathcal{L}']_k = (-1)^{1/2(k-1)}\Gamma_1 \qquad (k \text{ odd}) \tag{2.66}$$

$$[\mathcal{L},\mathcal{L}']_k = (-1)^{1/2(k-2)}\Gamma_2 \qquad (k \text{ even}) \tag{2.67}$$

and, of course,

$$[\mathcal{L},\mathcal{L}']_0 = \mathcal{L}' \tag{2.68}$$

where Γ_1 and Γ_2 are defined by

$$\Gamma_1 \equiv [\mathcal{L},\mathcal{L}'] = \hat{p}\cdot[\Omega\cdot\Omega'-\Omega'\cdot\Omega]\cdot\frac{\partial}{\partial\hat{p}} \tag{2.69}$$

and

$$\Gamma_2 = [\mathcal{L},\Gamma_1] = \hat{p}\cdot[\Omega'\cdot P+P\cdot\Omega']\cdot\frac{\partial}{\partial\hat{p}} \tag{2.70}$$

Equations (2.66) and (2.67) follow from the additional property

$$[\mathcal{L},\Gamma_2] = -\Gamma_1 \tag{2.71}$$

In Appendices D and E we prove that

$$\langle \Gamma_1(\vec{x},\hat{p})\mathcal{L}'(\vec{x},\hat{p})\rangle = -A(s)\mathcal{L} \tag{2.72}$$

and

$$\langle \Gamma_2(\vec{x},\hat{p})\mathcal{L}'(\vec{x},\hat{p})\rangle = A(s)(\mathcal{L}^2-\hat{\nabla}^2) \tag{2.73}$$

Thus, we can write

$$\begin{aligned}
\Delta_n = {} & (-1)^n\mathcal{L}^n \frac{1}{n!}\int_0^\infty dsA(s)s^n\hat{\nabla}_2\underline{f}_{00} + \\
& + (-1)^n \sum_{k=1,3,\ldots}^{n} \frac{(-1)^{1/2(k-1)}}{(n-k)!k!} \mathcal{L}^{n-k}\int_0^\infty dsA(s)s^n\mathcal{L}\underline{f}_{00} + \\
& - (-1)^n \sum_{k=2,4,\ldots}^{n} \frac{(-1)^{k/2}}{(n-k)!k!} \mathcal{L}^{n-k}\int_0^\infty dsA(s)s^n(\mathcal{L}^2-\hat{\nabla}^2)\underline{f}_{00}
\end{aligned} \tag{2.74}$$

Since $\mathcal{L}$ and $\hat{\nabla}^2$ commute, it is clear that each Δ_n commutes with $\hat{\nabla}^2$ and therefore the compatibility conditions are satisfied.

From Eqs. (2.25) and (2.29) we see that

$$\frac{\partial \underline{f}_0}{\partial \tau_2} = \sum_{n=0}^{\infty} \frac{\partial \underline{f}_{00}}{\partial \tau_{2n}}\epsilon^n = \sum_{n=0}^{\infty} \epsilon^n\Delta_n\underline{f}_0 \tag{2.75}$$

We substitute Eq. (2.74) into (2.75) and in the terms containing the summations over k , we first interchange the order of summation, then substitute for the summation over n , the summation over $m = n-k$ to obtain

$$\begin{aligned}
\frac{\partial \underline{f}_0}{\partial \tau_2} = {} & \int_0^\infty dsA(s)\sum_{n=0}^{\infty}(-\epsilon)^n\mathcal{L}^n \frac{1}{n!} s^n\hat{\nabla}^2\underline{f}_0 + \\
& + \int_0^\infty dsA(s)\sum_{k=1,3,\ldots}^{\infty} \frac{(-1)^{1/2(k-1)}}{k!}(-\epsilon s)^k \sum_{m=0}^{\infty}\frac{(-\epsilon)^m}{m!}\mathcal{L}^m s^m\mathcal{L}\underline{f}_0 + \\
& - \int_0^\infty dsA(s)\sum_{k=2,4,\ldots}^{\infty} \frac{(-1)^{k/2}}{k!}(-\epsilon s)^k \sum_{m=0}^{\infty}\frac{(-\epsilon)^m}{m!}\mathcal{L}^m s^m(\mathcal{L}^2-\hat{\nabla}^2)\underline{f}_0
\end{aligned} \tag{2.76}$$

The summations can now be recognized as the expansions of the exponential and trigonometric functions. Thus

$$\frac{\partial \underline{f}_0}{\partial \tau_2} = \int_0^\infty ds A(s) e^{-\epsilon \mathcal{L} s}[\cos \epsilon s \hat{\nabla}^2 + (1-\cos \epsilon s)\mathcal{L}^2 - \sin \epsilon s \mathcal{L}] \underline{f}_0 \qquad (2.77)$$

Along the restricted trajectory given by Eq. (2.38) we obtain to second order in $\epsilon\eta$

$$\frac{\partial \underline{f}_0}{\partial \tau}(\hat{p},\tau) + \epsilon \mathcal{L} f_0(\hat{p},\tau) = (\epsilon\eta)^2 \int_0^\infty ds A(s) e^{-\epsilon \mathcal{L} s}[\cos \epsilon s \hat{\nabla}^2 +$$

$$+ (1-\cos \epsilon s)\mathcal{L}^2 - \sin \epsilon s \mathcal{L}] f_0(\hat{p},\tau) \qquad (2.78)$$

Equation (2.78) gives a kinetic description of charged particles in a stationary random magnetic field. In the next section, the conditions under which this description is valid will be specified in more detail.

3. THE RELAXATION TO ISOTROPY

Equation (2.78) has the property of driving an arbitrary initial distribution function to isotropy. In fact, the final state is always the angular average in momentum of the initial state. To see this, we note that since $\mathcal{L}$ and $\hat{\nabla}^2$ commute, they must have common eigenfunctions. These eigenfunctions are the spherical harmonic functions with θ , the polar angle relative to the mean magnetic field and ϕ , the azimuthal angle about the mean field, as variables. We find with this choice

$$\hat{\nabla}^2 Y_{\ell m} = -\ell(\ell+1) Y_{\ell m} \qquad (3.1)$$

and

$$\mathcal{L} Y_{\ell m} = -im Y_{\ell m} \qquad (3.2)$$

We substitute an expansion of f_0 given by

$$f_0(\hat{p},\tau) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} f_{\ell m}(\tau) Y_{\ell m}(\theta,\phi) \tag{3.3}$$

into Eq. (2.78) and use the orthogonality of the spherical harmonics to obtain

$$\frac{\partial f_{\ell m}(\tau)}{\partial\tau} - i\epsilon m f_{\ell m}(\tau) =$$
$$= -(\epsilon\eta)^2\int_0^\infty ds A(s) e^{im\epsilon s}\Big[(\ell(\ell+1)-m^2)\cos\,\epsilon s + m^2 - im\,\sin\,\epsilon s\Big] f_{\ell m}(\tau) \tag{3.4}$$

The effect of the integral coefficient on the right side can be made more transparent by introducing the spatial Fourier transform of $A(s)$. We imagine a sensor moving through the stationary magnetic field with constant velocity, $\vec{U}$, measuring the magnitude of the components of the field orthogonal to $\vec{U}$. The sensor sees the random part of the field varying with time. Thus, the Fourier transform can be written as a power spectrum which depends on the angular frequency, kU/λ_n, where k is the dimensionless wave number. Then

$$A(s) = \frac{1}{\pi}\int_{-\infty}^{\infty} dk e^{iks}\,\frac{P_\perp\left(\frac{kU}{\lambda_n}\right)}{P_\perp(0)} \tag{3.5}$$

and Eq. (3.4) can be written

$$\frac{\partial f_{\ell m}}{\partial\tau} - i(\epsilon m - (\epsilon\eta)^2\Lambda'_{\ell m}) f_{\ell m} = -(\epsilon\eta)^2\Lambda_{\ell m} f_{\ell m}(\tau) \tag{3.6}$$

where

$$\Lambda_{\ell m} = m^2 \frac{P_\perp\left(\frac{\epsilon U}{\lambda_n} m\right)}{P_\perp(0)} + \frac{1}{2}(\ell(\ell+1)-m(m+1)) \frac{P_\perp\left(\frac{\epsilon U}{\lambda_n} (m+1)\right)}{P_\perp(0)} +$$

$$+ \frac{1}{2}(\ell(\ell+1)-m(m-1)) \frac{P_\perp\left(\frac{\epsilon U}{\lambda_n} (m-1)\right)}{P_\perp(0)} \quad (3.7)$$

and

$$\Lambda'_{\ell m} = \int_{-\infty}^{\infty} dk \frac{P_\perp\left(\frac{kU}{\lambda_n}\right)}{P_\perp(0)} \left[m^2\left(\frac{\mathscr{P}}{k+\epsilon m}\right)\right.$$

$$\left. + \frac{1}{2}(\ell(\ell+1)-m(m+1))\left(\frac{\mathscr{P}}{k+\epsilon(m+1)}\right) + \frac{1}{2}(\ell(\ell+1)-m(m-1))\left(\frac{\mathscr{P}}{k+\epsilon(m-1)}\right)\right] \quad (3.8)$$

in which $\mathscr{P}$ denotes the Cauchy principal value. Assuming a positive definite $P_\perp$, we see that $\Lambda_{\ell m}$ is always positive and $\Lambda_{00} = 0$. Thus, all $f_{\ell m}$ decay exponentially to zero with decay time (measured in units of the traversal time)

$$\tau_{\ell m} = \frac{1}{(\epsilon\eta)^2 \Lambda_{\ell m}} \quad (3.9)$$

except for f_{00} which remains constant. We notice also that the $f_{\ell m}$ oscillate with a frequency modified from the gyrofrequency in the mean field by the factor $(\epsilon\eta)^2\Lambda'_{\ell m}$. The difference between the real gyrofrequency and the gyrofrequency in the mean field depends on the specific power spectrum which is inserted into Eq. (3.8).

In order for the time scale extension technique which we have used to obtain this result to remain valid, the various time scales must remain distinct to preserve the ordering of terms in our power series expansions. Thus, the decay time must remain distinct from, and larger than, the traversal time. We must have

$\tau_{\ell m} \gg 1$. If $(\epsilon\eta)^2 \ll 1$, this condition is generally satisfied, unless f_0 is very anisotropic. In that case, we could have $\Lambda_{\ell m} \gg 1$ for large values of ℓ . Our kinetic theory is correct to second order in $(\epsilon\eta)$ only if the more restrictive condition

$$(\epsilon\eta)^2\Lambda_{\ell m} \ll 1 \tag{3.10}$$

is satisfied.

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APPENDIX A

We prove that

$$\langle \mathcal{L}'(\vec{x})\mathcal{L}'(\vec{x}')\rangle = A(r)\hat{\nabla}^2 \tag{A1}$$

when

$$\vec{x}' = \vec{x} - \hat{p}r \tag{A2}$$

where r is the magnitude of the separation between $\vec{x}$ and $\vec{x}'$

From the definition of $\mathcal{L}'$ we have

$$\langle \mathcal{L}'(\vec{x})\mathcal{L}'(\vec{x}')\rangle = -\frac{\partial}{\partial\hat{p}_i}\langle \Omega'(\vec{x})\cdot\hat{p}\hat{p}\cdot\Omega'(\vec{x}')\rangle_{ij}\frac{\partial}{\partial\hat{p}_j} \tag{A3}$$

but

$$\langle \Omega'(\vec{x})\cdot\hat{p}\hat{p}\cdot\Omega'(\vec{x}')\rangle_{ij} = \epsilon_{ik\ell}\epsilon_{mjn}\hat{p}_k\hat{p}_m R_{\ell n}(\vec{r}) \tag{A4}$$

where $\vec{r} = \vec{x}' - \vec{x}$ and where

$$R_{\ell n}(\vec{r}) = \langle \beta'_\ell(\vec{x})\beta'_n(\vec{x}')\rangle \tag{A5}$$

For the isotropic random field

$$R_{\ell n}(\vec{r}) = A(r)\delta_{\ell n} + B(r)r_\ell r_n \tag{A6}$$

where A and B are even functions of r . Notice that the second term, $B(r)r_\ell r_n$, in Eq. (A6) does not contribute to Eq. (A4) when $\vec{r} = \vec{x}' - \vec{x}$ is proportional to $\hat{p}$. Therefore

$$\begin{aligned}\langle \Omega'(\vec{x})\cdot\hat{p}\hat{p}\cdot\Omega'(\vec{x}')\rangle_{ij} &= \epsilon_{ik\ell}\epsilon_{mjn}\hat{p}_k\hat{p}_m A(r)\delta_{\ell n}\\ &= -(\delta_{ij}-\hat{p}_i\hat{p}_j)A(r)\,.\end{aligned} \tag{A7}$$

Thus

$$\langle \mathcal{L}'(\vec{x})\mathcal{L}'(\vec{x}')\rangle = A(r)\left(\frac{\partial}{\partial\hat{p}}\cdot n\cdot\frac{\partial}{\partial\hat{p}}\right) = A(r)\hat{\nabla}^2 \tag{A8}$$

where we have defined

$$\hat{\nabla}^2 \equiv \frac{\partial}{\partial\hat{p}}\cdot n\cdot\frac{\partial}{\partial\hat{p}} \tag{A9}$$

with

$$n_{ij} = \delta_{ij} - \hat{p}_i\hat{p}_j \tag{A10}$$

APPENDIX B

(a) We prove that

$$\pounds'\pounds^n = \sum_{k=0}^{n} \frac{(-1)^k n!}{(n-k)!k!} \pounds^{n-k}[\pounds,\pounds']_k \tag{B1}$$

where

$$[\pounds,\pounds']_k = (-1)^{1/2(k-1)}\Gamma_1 \quad (k \quad \text{odd}) \tag{B2}$$

$$[\pounds,\pounds']_k = (-1)^{1/2(k-2)}\Gamma_2 \quad (k \quad \text{even}) \tag{B3}$$

and

$$[\pounds,\pounds']_0 = \pounds' \tag{B4}$$

The proof follows by noting that for $n = 0$ Eq. (B1) is an obvious identity and for $n = 1$ and $n = 2$ Eq. (B1) is also true if we remember the definitions of Γ_1 and Γ_2

$$\Gamma_1 = [\pounds,\pounds'] , \qquad \Gamma_2 = [\pounds,\Gamma_1] \tag{B5}$$

We show that the $(n+1)$'th term given by Eq. (B1) is in agreement with the results of operating from the right on Eq. (B1) with $\pounds$. Thus, any order can be generated from the $n=0$ and 1 cases.

From Eq. (B1)

$$\pounds'\pounds^{n+1} = \sum_{k=0}^{n} \frac{(-1)^k n!}{(n-k)!k!} \pounds^{n-k}[\pounds,\pounds']_k\pounds \tag{B6}$$

Let n be an odd integer. Then

$$\mathcal{L}'\mathcal{L}^{n+1} = \mathcal{L}^n\mathcal{L}'\mathcal{L} +$$

$$+\sum_{k=1,3\ldots}^{n} \frac{(-1)^k n!}{(n-k)!k!} \mathcal{L}^{n-k}(-1)^{1/2(k-1)}\Gamma_1\mathcal{L} +$$

$$+\sum_{k=2,4,\ldots}^{n-1} \frac{(-1)^k n!}{(n-k)!k!} \mathcal{L}^{n-k}(-1)^{1/2(k-2)}\Gamma_2\mathcal{L} \qquad \text{(B7)}$$

We define $m = n + 1$ and commute the $\mathcal{L}'\Gamma_1$ and Γ_2 which appear in Eq. (B7) to obtain

$$\mathcal{L}'\mathcal{L}^m = \mathcal{L}^m\mathcal{L}' - \mathcal{L}^{m-1}[\mathcal{L},\mathcal{L}']_1 +$$

$$+\sum_{k=1,3,\ldots}^{m-1} \frac{(-1)^k(m-1)!}{(m-1-k)!k!} (-1)^{1/2(k-1)}\mathcal{L}^{m-1-k}(\mathcal{L}\Gamma_1-\Gamma_2) +$$

$$+\sum_{k=2,4,\ldots}^{m-2} \frac{(-1)^k(m-1)!}{(m-1-k)!k!} (-1)^{1/2(k-2)}\mathcal{L}^{m-1-k}(\mathcal{L}\Gamma_2+\Gamma_1) \qquad \text{(B8)}$$

With $\kappa = k + 1$, Eq. (B8) can be written

$$\mathcal{L}'\mathcal{L}^m = \mathcal{L}^m\mathcal{L}' - \mathcal{L}^{m-1}[\mathcal{L},\mathcal{L}']_1 + \sum_{k=1,3,\ldots}^{m-1} \frac{(-1)^k(m-1)!}{(m-1-k)!k!} \mathcal{L}^{m-k}[\mathcal{L},\mathcal{L}']_k +$$

$$-\sum_{\kappa=2,4,\ldots}^{m} \frac{(-1)^{\kappa-1}(m-1)!}{(m-\kappa)!(\kappa-1)!} (-1)^{1/2(\kappa-2)}\mathcal{L}^{m-\kappa}\Gamma_2 +$$

$$+\sum_{k=2,4,\ldots}^{m-2} \frac{(-1)^k(m-1)!}{(m-1-k)!k!} \mathcal{L}^{m-k}[\mathcal{L},\mathcal{L}']_k +$$

$$+\sum_{\kappa=3,5,\ldots}^{m-1} \frac{(-1)^{\kappa-1}(m-1)!}{(m-\kappa)!(\kappa-1)!} \mathcal{L}^{m-\kappa}(-1)^{1/2(\kappa-3)}\Gamma_1 \qquad \text{(B9)}$$

Now, letting $\kappa = k$ and combining the summations over odd and even values of k , we find

$$\pounds'\pounds^m = \pounds^m\pounds' + \sum_{k=1,3,\ldots}^{m-1} \frac{(-1)^k m!}{(m-k)!k!} \pounds^{m-k}[\pounds,\pounds']_k + \sum_{k=2,4,\ldots}^{m-2} \frac{(-1)^k m!}{(m-k)!k!} \pounds^{m-k}[\pounds,\pounds']_k \tag{B10}$$

or

$$\pounds'\pounds^m = \sum_{k=0}^{m} \frac{(-1)^k m!}{(m-k)!k!} \pounds^{m-k}[\pounds,\pounds']_k \tag{B11}$$

which completes the proof for odd n . The proof for even n is similar.

(b) We prove that

$$[\mathcal{L},\mathcal{L}']_n = \sum_{k=0}^{n} \frac{(-1)^k n!}{(n-k)!k!} \mathcal{L}^{n-k}\mathcal{L}'\mathcal{L}^k \tag{B12}$$

for all n by noting that for $n = 0$ and 1, Eq. (B12) is obviously true, and that given Eq. (B12) for an arbitrary n, then the correct expression for the $(n+1)$'th term is generated by commuting Eq. (B12) with $\bar{\mathcal{L}}$. Thus

$$[\mathcal{L},\mathcal{L}']_{n+1} = \sum_{k=0}^{n} \frac{(-1)^k n!}{(n-k)!k!} [\mathcal{L},\mathcal{L}^{n-k}\mathcal{L}'\mathcal{L}^k] \tag{B13}$$

Define $m = n + 1$. Then Eq. (B13) is given by

$$\begin{aligned}[\mathcal{L},\mathcal{L}']_m &= \sum_{k=0}^{m-1} \frac{(-1)^k (m-1)!}{(m-1-k)!k!} \mathcal{L}^{m-k}\mathcal{L}'\mathcal{L}^k + \\ &+ \sum_{k=1}^{m} \frac{(-1)^k (m-1)!}{(m-k)!(k-1)!} \mathcal{L}^{m-k}\mathcal{L}'\mathcal{L}^k \end{aligned} \tag{B14}$$

where the summation index in the second term on the right side has been shifted by one. The summations on the right side of Eq. (B14) for $k = 1$ to $m - 1$ can be combined to give

$$[\mathcal{L},\mathcal{L}']_m = \mathcal{L}^m\mathcal{L}' + (-1)^m\mathcal{L}'\mathcal{L}^m + \sum_{k=1}^{m-1} \frac{(-1)^k m!}{(m-k)!k!} \mathcal{L}^{m-k}\mathcal{L}'\mathcal{L}^k \tag{B15}$$

which can be written

$$[\mathcal{L},\mathcal{L}']_m = \sum_{k=0}^{m} \frac{(-1)^k m!}{(m-k)!k!} \mathcal{L}^{m-k}\mathcal{L}'\mathcal{L}^k \tag{B16}$$

and which completes our proof.

APPENDIX C

The Γ-operators are defined by the commutators

$$[\mathcal{L},\mathcal{L}'] \equiv \Gamma_1 \tag{C1}$$

$$[\mathcal{L},\Gamma_1] \equiv \Gamma_2 \tag{C2}$$

In addition,

$$[\mathcal{L},\Gamma_2] = -\Gamma_1 \tag{C3}$$

To establish notation, we rewrite Eqs. (C1), (C2) and (C3) in the following manner

$$[\mathcal{L},\mathcal{L}']_1 \equiv [\mathcal{L},\mathcal{L}'] = \Gamma_1 \tag{C4}$$

$$[\mathcal{L},\mathcal{L}']_2 \equiv [\mathcal{L},[\mathcal{L},\mathcal{L}']] = \Gamma_2 \tag{C5}$$

$$[\mathcal{L},\mathcal{L}']_3 \equiv [\mathcal{L},[\mathcal{L},[\mathcal{L},\mathcal{L}']]] = -\Gamma_1 \tag{C6}$$

In this appendix, we prove that

$$[\mathcal{L},\mathcal{L}']_n = (-1)^{1/2(n-1)}\Gamma_1 \qquad (n \ \text{odd}) \tag{C7}$$

and

$$[\mathcal{L},\mathcal{L}']_n = (-1)^{1/2(n-2)}\Gamma_2 \qquad (n \ \text{even}) \tag{C8}$$

Notice that Eqs. (C7) and (C8) reduce to Eqs. (C4) and (C5) for $n = 1$ and 2. To prove Eqs. (C7) and (C8), we assume them valid for the N'th term and prove that the (N+1)'th and (N+2)'th term are generated by the commutation rules of Eqs. (C2) and (C3).

Let $n = N$ be an odd integer. Assume

$$[\mathcal{L},\mathcal{L}']_N = (-1)^{1/2(N-1)}\Gamma_1 \tag{C9}$$

Then

$$[\mathcal{L},\mathcal{L}']_{N+1} = (-1)^{1/2(N-1)}[\mathcal{L},\Gamma_1]$$

$$= (-1)^{1/2(N-1)}\Gamma_2 \tag{C10}$$

Define $M \equiv N+1$, an even integer. From Eq. (C10)

$$[\mathcal{L},\mathcal{L}']_M = (-1)^{1/2(M-2)}\Gamma_2 \tag{C11}$$

in agreement with Eq. (C8).

Now

$$[\mathcal{L},\mathcal{L}']_{M+1} = (-1)^{1/2(M-2)}[\mathcal{L},\Gamma_2] \tag{C12}$$

From Eq. (C3)

$$[\mathcal{L},\mathcal{L}']_{M+1} = (-1)^{1/2M}\Gamma_1 \tag{C13}$$

or

$$[\mathcal{L},\mathcal{L}']_{N+2} = (-1)^{1/2[(N+2)-1]}\Gamma_1 \tag{C14}$$

Thus, all odd n terms can be generated starting from Eq. (C4). The proof of the even n terms is similar.

APPENDIX D

We prove that

$$\langle \Gamma_1(\vec{x},\hat{p})\mathcal{L}'(\vec{x}',\hat{p})\rangle = -A(r)\mathcal{L} \tag{D1}$$

when

$$\vec{x}' = \vec{x} - \hat{p}r \tag{D2}$$

The operator Γ_1 can be written in the form

$$\Gamma_1 = \hat{p}\cdot(\vec{\beta}'\hat{\beta}-\hat{\beta}\vec{\beta}')\cdot\frac{\partial}{\partial\hat{p}} \tag{D3}$$

Therefore, Eq. (D1) is given by

$$-\langle \hat{p}_i(\beta_i'(\vec{x})\beta_j-\beta_i\beta_j'(\vec{x}))\frac{\partial}{\partial\hat{p}_j}\hat{p}_k\epsilon_{k\ell m}\beta_m'(\vec{x}')\frac{\partial}{\partial\hat{p}_\ell}\rangle \tag{D4}$$

which can be rewritten as

$$-\hat{p}_i\frac{\partial}{\partial\hat{p}_j}\hat{p}_k\epsilon_{k\ell m}\left[\beta_jR_{im}(\vec{r})-\beta_iR_{jm}(\vec{r})\right]\frac{\partial}{\partial\hat{p}_\ell} \tag{D5}$$

where $\vec{r} = \vec{x}' - \vec{x}$. We assume the isotropic random field

$$R_{ij}(\vec{r}) = A(r)\delta_{ij} + B(r)r_ir_j \tag{D6}$$

Then Eq. (D5) reduces to

$$-\hat{p}_i\frac{\partial}{\partial\hat{p}_j}\hat{p}_k\epsilon_{k\ell m}\left[(\delta_{im}A+r^2B\hat{p}_i\hat{p}_m)\beta_j = -\beta_i(\delta_{jm}A+r^2B\hat{p}_j\hat{p}_m)\right]\frac{\partial}{\partial\hat{p}_\ell} \tag{D7}$$

Because of the skew-symmetric $\epsilon_{k\ell m}$ the terms containing $B(r)$ do not contribute. Some further manipulation of Eq. (D7) leads to

$$-\hat{p}_i\left[(\delta_{jk}-\hat{p}_j\hat{p}_k) + \hat{p}_k \frac{\partial}{\partial\hat{p}_j}\right]\epsilon_{k\ell i}\beta_j \frac{\partial}{\partial\hat{p}_\ell} A +$$

$$+\hat{p}_i\left[(\delta_{jk}-\hat{p}_j\hat{p}_k) + \hat{p}_k \frac{\partial}{\partial\hat{p}_j}\right]\epsilon_{k\ell j}\beta_i \frac{\partial}{\partial\hat{p}_\ell} A \tag{D8}$$

Again, because of the skew-symmetric ϵ, only the first term of Eq. (D7) contributes. Thus,

$$\begin{aligned}\langle\Gamma_1(\vec{x},\hat{p})\mathcal{L}'(\vec{x},\hat{p})\rangle &= A(r)\hat{p}_i\epsilon_{i\ell j}\beta_j \frac{\partial}{\partial p_\ell} \\ &= -A(r)\mathcal{L}\end{aligned} \tag{D9}$$

APPENDIX E

We prove that

$$\langle \Gamma_2(\vec{x},\hat{p})\mathcal{L}'(\vec{x}',\hat{p})\rangle = A(r)(\mathcal{L}^2-\hat{\nabla}^2) \tag{E1}$$

when $\vec{x}' = \vec{x} - \hat{p}r$ by substituting in Eq. (E1) the following expression for Γ_2

$$\Gamma_2 = (\beta\cdot\beta')\mathcal{L} - \mathcal{L}' \tag{E2}$$

Thus

$$\begin{aligned}\langle \Gamma_2(\vec{x},\hat{p})\mathcal{L}'(\vec{x}',\hat{p})\rangle &= \mathcal{L}\langle(\hat{\beta}\cdot\vec{\beta}'(\vec{x}))\mathcal{L}'(\vec{x}',\hat{p})\rangle - \langle\mathcal{L}'(\vec{x},\hat{p})\mathcal{L}'(\vec{x}',\hat{p})\rangle \\ &= \mathcal{L}\langle(\hat{\beta}\cdot\vec{\beta}'(\vec{x}))\mathcal{L}'(\vec{x}',\hat{p})\rangle - A(r)\hat{\nabla}^2\end{aligned} \tag{E3}$$

We must evaluate

$$\begin{aligned}\langle(\hat{\beta}\cdot\vec{\beta}'(\vec{x}))\mathcal{L}'(\vec{x}',\hat{p})\rangle &= -\beta_m\langle\beta_m'(\vec{x})\hat{p}_i\epsilon_{ijk}\beta_k'(\vec{x}')\frac{\partial}{\partial\hat{p}_j}\rangle \\ &= -\beta_m p_i\epsilon_{ijk}R_{mk}(\vec{r})\frac{\partial}{\partial\hat{p}_j}\end{aligned} \tag{E4}$$

where $\vec{r} = \vec{x}' - \vec{x}$. With

$$R_{mk}(\vec{r}) = A(r)\delta_{mk} + r^2B(r)\hat{p}_m\hat{p}_k \tag{E5}$$

we find

$$\langle(\hat{\beta}\cdot\vec{\beta}'(\vec{x}))\mathcal{L}'(\vec{x}',\hat{p})\rangle = \mathcal{L}A(r) \tag{E6}$$

Substituting Eq. (E6) into Eq. (E3), we obtain Eq. (E1)

CHAPTER III

OMNIDIRECTIONAL INTENSITY AND FLUX EQUATIONS FOR ALL BUT VERY LOW ENERGIES

INTRODUCTION

Most modulation phenomena occur in the interplanetary magnetic field in the vicinity $\epsilon \gtrsim 1$. Thus, expansions in ϵ, as presented in the previous chapter, are not well suited for studies of the modulation problem. Expansions in $1/\epsilon$ might be somewhat more useful; but, as we have already noted, such expansions have been carried out incorrectly until the present time.

In this chapter, we present an alternate method for obtaining a useful modulation theory. We treat $\epsilon \sim 1$ right from the start carrying the full dependence on ϵ in the truncated master equation. To reduce the complexity of the resulting theory, we give up a full kinetic description of the cosmic ray distribution function by studying only the first two moments of a distribution function which is assumed to be very nearly isotropic. This approach is well suited to studies of the observed cosmic ray distribution function where anisotropies have been found to be very small.

1. THE OMNIDIRECTIONAL INTENSITY AND FLUX EQUATIONS

We expand the distribution function about isotropy in terms of the spherical harmonic functions. The result is

$$p^2 f = \frac{1}{4\pi} I + \frac{3}{4\pi} \hat{p} \cdot \vec{\phi} + \ldots \tag{1.1}$$

where $I(\tau)$ is the omnidirectional intensity given by

$$I(\tau) = p^2 \int d\Omega_p f(\hat{p},\tau) \tag{1.2}$$

and $\vec{\phi}(\tau)$ is the flux given by

$$\vec{\phi}(\tau) = p^2 \int d\Omega_p \hat{p} f(\hat{p},\tau) \tag{1.3}$$

The integrations in Eqs. (1.2) and (1.3) are carried out over all directions on the unit momentum sphere. Notice that both I and $\vec{\phi}$ are differential functions of rigidity. The next term in the expansion in Eq. (1.1) measures the anisotropy in the cosmic ray pressure tensor, which we neglect.

By substituting Eq. (1.1) into the truncated master equation and using the orthogonality of the spherical harmonics on the unit momentum sphere, we find

$$\frac{\partial I}{\partial \tau} = 0 \tag{1.4}$$

and

$$\frac{\partial \vec{\phi}}{\partial \tau} - \epsilon \Omega \cdot \vec{\phi} = \frac{3}{4\pi} (\epsilon\eta)^2 \int d\Omega_p \int_0^\tau d\lambda \langle\!\langle (\Omega' \cdot \hat{p}) e^{-\mathcal{H}\lambda} (\hat{p} \cdot \Omega') \rangle\!\rangle \cdot \vec{\phi}(\tau-\lambda) \tag{1.5}$$

We must find the behavior of $\vec{\phi}(\tau)$ for large τ and then, on adding weak spatial gradients in I , we can find the transport equation for I which is appropriate for the modulation problem. The major simplification which is introduced by the study of

only the lowest moments of f is readily apparent from Eq. (1.5). Equation (1.5) is in the form of a matrix exponential equation for $\vec{\phi}$ where the integral operator which operates on $\vec{\phi}(\tau-\lambda)$ is no longer a differential operator in $\hat{p}$. In addition, because of the integration over Ω_p which remains, only certain low-order moments of the operator $\langle(\Omega'\cdot\hat{p})e^{-\mathcal{H}\lambda}(\hat{p}\cdot\Omega')\rangle$ contribute to the evolution of $\vec{\phi}$; the full complexity of this operator never enters.

In the "interaction" representation of Eq. (1.5), we have for

$$\vec{S}(\tau) = e^{-\epsilon\Omega\tau}\cdot\vec{\phi}(\tau) \tag{1.6}$$

the equation

$$\frac{\partial\vec{S}}{\partial\tau} = \frac{3}{4\pi}(\epsilon\eta)^2\int d\Omega_p \int_0^\tau d\lambda\langle(\Omega'\cdot\hat{p})\hat{p}\cdot\tilde{\Omega}'(\vec{x}-\vec{r})\rangle\cdot\vec{S}(\tau-\lambda) \tag{1.7}$$

where

$$\tilde{\Omega}'(\vec{x}-\vec{r}) = \Omega\left(\hat{\beta}\cdot\vec{\beta}'(\vec{x}-\vec{r})\right) - \Gamma_2(\vec{x}-\vec{r})\cos\,\epsilon\lambda + \Gamma_1(\vec{x}-\vec{r})\sin\,\epsilon\lambda \tag{1.8}$$

and where

$$\vec{r} = \left[P + N\left(\frac{\sin\,\epsilon\lambda}{\epsilon\lambda}\right) + \Omega\left(\frac{\cos\,\epsilon\lambda-1}{\epsilon\lambda}\right)\right]\cdot\hat{p}\lambda \tag{1.9}$$

To take advantage of the fact that Eq. (1.7) only depends on certain low-order moments of the operator $\langle(\Omega'\cdot\hat{p})\hat{p}\cdot\tilde{\Omega}'(\vec{x}-\vec{r})\rangle$, we introduce Legendre polynomial expansions of $A(r)$ and $B(r)$ from the correlation function. We let

$$A(r) = \sum_{\ell=0}^{\infty} A_\ell(\lambda)P_\ell(z) \tag{1.10}$$

and

$$B(r) = \sum_{\ell=0}^{\infty} B_\ell(\lambda)P_\ell(z) \tag{1.11}$$

where $P_\ell(z)$ is the ℓ'th-order Legendre polynomial function

of $z = (\hat{p}\cdot\hat{\beta})$ and where r is the magnitude of Eq. (1.9) given by

$$r = \lambda\left[1 + (1-z^2)\left(\frac{1 - \frac{1}{2}(\epsilon\lambda)^2 - \cos\epsilon\lambda}{\frac{1}{2}(\epsilon\lambda)^2}\right)\right]^{1/2} \tag{1.12}$$

The coefficients $A_\ell(\lambda)$ and $B_\ell(\lambda)$ are given by

$$A_\ell(\lambda) = \left(\frac{2\ell+1}{2}\right)\int_{-1}^{1} A(r)P_\ell(z)dz \tag{1.13}$$

and

$$B_\ell(\lambda) = \left(\frac{2\ell+1}{2}\right)\int_{-1}^{1} B(r)P_\ell(z)dz \tag{1.14}$$

After considerable algebra, we find

$$\begin{aligned}
\frac{\partial\vec{S}}{\partial\tau} = &- (\epsilon\eta)^2 \frac{3}{4\pi}\int_0^\tau d\lambda\left[2J_\perp\cos\epsilon\lambda - 8\lambda^2K^{(1)}\left(\frac{\cos\epsilon\lambda-1}{\epsilon\lambda}\right)^2\right]P\cdot\vec{S}(\tau-\lambda) + \\
&- (\epsilon\eta)^2 \frac{3}{4\pi}\int_0^\tau d\lambda\left[J_\perp + J_\parallel\cos\epsilon\lambda - \lambda^2(K^{(1)}+K^{(2)})\left(2\frac{\sin\epsilon\lambda}{\epsilon\lambda}\left(1 - \frac{\sin\epsilon\lambda}{\epsilon\lambda}\right) + \right.\right. \\
&\left.\left. - 2\left(\frac{\cos\epsilon\lambda-1}{(\epsilon\lambda)^2}\right) - 1\right)\right]N\cdot\vec{S}(\tau-\lambda) + \\
&+ (\epsilon\eta)^2 \frac{3}{4\pi}\int_0^\tau d\lambda\left[J_\parallel\sin\epsilon\lambda - \lambda^2(K^{(1)}+K^{(2)})\,\cdot\right. \\
&\left.\cdot\left(2\left(\frac{\cos\epsilon\lambda-1}{\epsilon\lambda}\right)\left(1 - \frac{\sin\epsilon\lambda}{\epsilon\lambda}\right)\right)\right]\Omega\cdot\vec{S}(\tau-\lambda)
\end{aligned} \tag{1.15}$$

where

$$J_{\perp} = \frac{4\pi}{3}\left[A_0 - \frac{1}{5} A_2\right] \tag{1.16}$$

$$J_{\parallel} = \frac{4\pi}{3}\left[\frac{2}{5} A_2 + A_0\right] \tag{1.17}$$

$$K^{(1)} = \frac{4\pi}{15}\left[\frac{1}{21} B_4 - \frac{2}{7} B_2 + B_0\right] \tag{1.18}$$

and

$$K^{(1)} + K^{(2)} = \frac{4\pi}{15}\left[-\frac{4}{21} B_4 + \frac{4}{7} B_2 + B_0\right] \tag{1.19}$$

2. THE LIMIT, $\epsilon = 0$

In the limit $\epsilon = 0$ (but $(\epsilon\eta)^2$ finite) we find

$$K^{(1)} = \frac{4\pi}{15} B_0 \quad , \qquad K^{(1)} + K^{(2)} = \frac{4\pi}{15} B_0 \tag{2.1}$$

and

$$J_{\parallel} = \frac{4\pi}{3} A_0 \quad , \qquad J_{\perp} = \frac{4\pi}{3} A_0 \tag{2.2}$$

In addition,

$$A_0 = A(\lambda) \quad \text{and} \quad B_0 = B(\lambda) \tag{2.3}$$

Therefore

$$\frac{\partial \vec{S}}{\partial \tau} = -2(\epsilon\eta)^2 \int_0^\tau d\lambda \; A(\lambda) \; \vec{S}(\tau-\lambda) \tag{2.4}$$

In this limit, $\vec{\phi}(\tau)$ obeys the same equation as $\vec{S}(\tau)$.

A straightforward linear time-scale extension of Eq. (2.4) yields

$$\frac{\partial \vec{\phi}_0}{\partial \tau} = -\vec{\phi}_0 \tag{2.5}$$

where $\vec{\phi}_0$ is the leading term in a power series expansion of $\vec{\phi}$ in powers of $(\epsilon\eta)^2$. We have, in addition, normalized all lengths to the parallel integral length

$$\lambda_p = 2 \int_0^\infty d\lambda \; A(\lambda) \tag{2.6}$$

Upon adding weak spatial gradients, we find

$$\frac{\partial I}{\partial \tau} + \alpha \vec{\nabla} \cdot \vec{\phi}_0 = 0 \tag{2.7}$$

and

$$\frac{\partial \vec{\phi}_0}{\partial \tau} + \vec{\phi}_0 = - \frac{1}{3} \alpha \vec{\nabla} I \tag{2.8}$$

In these equations, the gradients are measured in units of L, a macroscopic length which measures the size of the system containing the particles and

$$\alpha = \frac{\lambda_p}{L} \tag{2.9}$$

For $I(\tau)$ slowly varying in τ, we can eliminate $\vec{\phi}_0$ from Eqs. (2.6) and (2.7) when $\tau \gtrsim 1/\alpha$. The result is

$$\frac{\partial I_0}{\partial T} + \alpha \vec{\nabla} \cdot (\vec{V} I_0) = \frac{1}{3} \frac{\alpha}{(\epsilon\eta)^2} \nabla^2 I_0 \tag{2.10}$$

a familiar convection-diffusion transport equation for I_0, the leading term in a power series expansion of I in α. A new time scale has been introduced in Eq. (2.10)

$$T = \alpha\tau \tag{2.11}$$

The velocity, $\vec{V}$, is the average Alfvén drift velocity due to curvature and gradients in the magnetic field averaged over an isotropic distribution of particles. This result is in agreement with the previous chapter if the expansion of Eq. (1.1) is substituted into the high-energy theory calculated there.

3. FINITE ENERGY $(\epsilon \sim 1)$

Equation (1.15) can be written in the general form

$$\frac{\partial \vec{S}}{\partial \tau} = - (\epsilon\eta)^2 \int_0^\tau d\lambda K(\epsilon,\lambda)\cdot\vec{S}(\tau-\lambda) \qquad (3.1)$$

where $K(\epsilon,\lambda)$ is a tensor kernel which symbolizes the large collection of terms shown in Eq. (1.15). We have seen from Eq. (2.4) that when $\epsilon \to 0$, $K(\epsilon,\lambda)$ takes on a very simple form with the attractive feature that it has a sharp cutoff in λ due to the finite range of the correlation coefficient, $A(\lambda)$. It is this property of the finite range of $K(\epsilon,\lambda)$ when $\epsilon \to 0$ that allows us to apply the linear time scale analysis to obtain the high-energy kinetic and transport theories. When $\epsilon \sim 1$, however, the kernel takes on a long tail which goes as $1/\lambda$. The mathematical techniques for handling Eq. (3.1) with such a kernel have not been developed previously. In the next two chapters, we develop the necessary techniques to determine the long-time behavior of Eq. (3.1) when the kernel is long-ranged.

The long tail in $K(\epsilon,\lambda)$ is a general feature which comes from those particles which have pitch angles relative to the mean field close to 90°. The basic interaction between a charged particle and the random magnetic field lasts as long as the particle remains within a length λ_p of its starting position at some arbitrary origin in time when the interaction is assumed to start. When $\epsilon \ll 1$, the gyroradius of the particle is very large compared to λ_p. Therefore, the particle's trajectory is essentially straight through the correlated region; the distance between the particle's position at time t and at time $t=0$ grows linearly with time, and the length, λ_p, is quickly achieved. On the other hand, when $\epsilon \sim 1$, the gyroradius and λ_p are equal in magnitude. Therefore, we must take into account the helical nature of the particle's trajectory in the mean field.

If the velocity of the particle parallel to the mean field is small, the particle may take a long time to get further than λ_p from its starting point. The particle-field interaction can last very long and, as a result, can contribute an inordinately large amount to the interaction integral operator on the right side of Eq. (3.1). The result is $K(\lambda) \propto 1/\lambda$ for $\lambda >> 1$. We present, here, a specific demonstration of how one of the coefficients involved develops the long tail for a Gaussian correlation function. By examining any one of the A_ℓ's or B_ℓ's one can convince oneself that the results shown here are a general feature of $\epsilon \sim 1$ even though $A(r)$ and $B(r)$ may be short-ranged functions.

We calculate $A_0(\lambda,\epsilon)$ when

$$A(r) = e^{-r^2} \tag{3.2}$$

We define the function

$$\zeta(\epsilon\lambda) = \frac{1 - \frac{1}{2}(\epsilon\lambda)^2 - \cos \epsilon\lambda}{\frac{1}{2}(\epsilon\lambda)^2} \tag{3.3}$$

Then

$$r = \lambda[(1+\zeta(\epsilon\lambda)) - \zeta(\epsilon\lambda)z^2]^{1/2} \tag{3.4}$$

Therefore

$$A_0(\lambda,\epsilon) = e^{-\lambda^2(1+\zeta(\epsilon\lambda))}\int_0^1 dz\ e^{-\lambda^2\zeta(\epsilon\lambda)z^2}$$

$$= \sqrt{\frac{\pi}{4}}\,\frac{e^{-\lambda^2(1+\zeta(\epsilon\lambda))}}{\lambda\sqrt{\zeta(\epsilon\lambda)}}\,\mathrm{Erf}(\lambda\sqrt{\zeta(\epsilon\lambda)}) \tag{3.5}$$

Notice, for $\epsilon = 0$, $A_0(\lambda,\epsilon) = e^{-\lambda^2}$. However, for finite ϵ and for large $\epsilon\lambda$,

$$A_0(\lambda,\epsilon) \underset{\epsilon\lambda \gg 1}{\rightarrow} \sqrt{\frac{\pi}{4}}\, \frac{e^{-\frac{2}{\epsilon^2}(1 - \cos \epsilon\lambda)}}{\lambda} \tag{3.6}$$

In the next two chapters, we study models of Eq. (3.1) to develop the mathematical techniques necessary to find the long-time behavior of the flux. In the first of these chapters, we consider a class of matrix exponential equations in which Eq. (3.1) is contained. In the following chapter, we study a scalar model of Eq. (3.1) with a long-range kernel given by

$$K(\lambda) = \frac{1}{1 + \lambda} \tag{3.7}$$

We will see that the behavior of functions governed by this kernel is rather surprising; in particular, the decay time is not analytic in the expansion parameter but behaves as $(\epsilon |\ell n\ \epsilon|)^{-1}$.

CHAPTER IV

APPROXIMATION METHODS FOR THE GOVERNING EQUATIONS

INTRODUCTION

We discuss here the uniform expansion of matrix exponentials in a rather general form. We succeed in obtaining uniformizing formulae under the conditions that the generator of the time translations is a linear operator in Hilbert space. This means that we are not confining ourselves to a finite dimensional matrix system. The major restriction in the analysis is the assumption that the direct Taylor expansion of the unknown function has secular behavior which is of a polynomial nature. The polynomial has as its maximum degree the order of the term in the direct perturbation expansion. This restriction is serious and effort is being made to remove it. It is, however, worth noting that with the definite polynomial assumption made very definite conclusions can be reached with practically no restrictions on the operators involved. Thus, a class of partial differential equations which is considerably wide is included. The type of equation included, in fact, encompasses the class discussed by Akhiezer in his large monograph.

1. NONDEGENERATE PERTURBATION THEORY

We write the equation to be solved in terms of a Schrodinger form as

$$i \frac{\partial U}{\partial \tau} + AU = -\delta BU \tag{1.1}$$

We decompose the perturbation as

$$B = \alpha A + \Gamma \tag{1.2}$$

where we have chosen

$$\mathrm{Tr}(A \cdot \Gamma) = 0 \tag{1.3}$$

We readily obtain

$$\alpha = \frac{\mathrm{Tr}(A \cdot B)}{\mathrm{Tr}(A \cdot A)} \tag{1.4}$$

Substitution of (1.2) into (1.1) yields

$$i \frac{\partial U}{\partial \tau} + (1+\alpha\delta)A \cdot U = -\delta\Gamma \cdot U \tag{1.5}$$

We redefine the parameters of relevance by introducing

$$s = (1+\alpha\delta)\tau \quad , \qquad \epsilon = \frac{\delta}{1 + \alpha\delta} \tag{1.6}$$

so that the original Schrodinger equation now reads

$$i \frac{\partial U}{\partial s} + A \cdot U = -\epsilon\Gamma \cdot U \tag{1.7}$$

We now proceed to perform an expansion of the following form

$$s(\tau_0) = \tau_0 + \epsilon\xi_0(\tau_0) + \epsilon^2\eta_0(\tau_0) + \dots \tag{1.8}$$

$$\tau_1(\tau_0) = \qquad \epsilon\xi_1(\tau_0) + \epsilon^2\eta_1(\tau_0) + \dots \tag{1.9}$$

$$\tau_2(\tau_0) = \qquad\qquad \epsilon^2\eta_2(\tau_0) + \dots \tag{1.10}$$

We readily obtain for the derivative with respect to the parameter, s ,

$$\frac{\partial}{\partial s} = \left(\frac{ds}{d\tau_0}\right)^{-1}\left[\frac{\partial}{\partial\tau_0} + \epsilon\dot{\xi}_1\frac{\partial}{\partial\tau_1} + \epsilon^2\left(\dot{\eta}_1\frac{\partial}{\partial\tau_1} + \dot{\eta}_2\frac{\partial}{\partial\tau_2}\right) + \ldots\right] \tag{1.11}$$

Similarly, we have

$$\frac{ds}{d\tau_0} = 1 + \epsilon\dot{\xi}_0 + \epsilon^2\dot{\eta}_0 + \ldots \tag{1.12}$$

Thus, the Schrodinger equation reads

$$i\left[\frac{\partial}{\partial\tau_0} + \epsilon\dot{\xi}_1\frac{\partial}{\partial\tau_1} + \epsilon^2\left(\dot{\eta}_1\frac{\partial}{\partial\tau_1} + \dot{\eta}_2\frac{\partial}{\partial\tau_2}\right) + \ldots\right]U =$$

$$= -(1 + \epsilon\dot{\xi}_0 + \epsilon^2\dot{\eta}_0 + \ldots)(A + \epsilon\Gamma)U \tag{1.13}$$

We notice that the expansion used (represented by Eqs. (1.8)-(1.10)) is sufficiently general to include both Lighthill stretching and a time scale. In particular, many time scales are included.

In zeroth order, we find

$$i\frac{\partial U_0}{\partial\tau_0} + AU_0 = 0 \tag{1.14}$$

whose solution can be written as

$$U_0(\vec{\tau}_0) = e^{iA\tau_0}\cdot U_0(\vec{\tau}_1) \tag{1.15}$$

A first-order equation can be written as

$$i\frac{\partial U_1}{\partial\tau_0} + A\cdot U_1 = -i\dot{\xi}_1\frac{\partial U_0}{\partial\tau_1} - \dot{\xi}_0 A\cdot U_0 - \Gamma\cdot U_0 \tag{1.16}$$

We can solve this equation as

$$U_1(\vec{\tau}_0) = e^{iA\tau_0}\cdot U_1(\vec{\tau}_1) - \xi_1\frac{\partial U_0}{\partial\tau_1} + i\xi_0 A\cdot U_0 + i\int_0^{\tau_0} d\lambda\widetilde{\Gamma}(\lambda)\cdot U_0 \tag{1.17}$$

where we have introduced the convenient abbreviation

$$\widetilde{\Gamma}(\lambda) = e^{iA\lambda}\cdot\Gamma\cdot e^{-iA\lambda} \tag{1.18}$$

In order to study the secularity, it is useful to multiply the equation (1.17) by the inverse of the leading-order operator as given by (1.15). We then find

$$U_1(\vec{\tau}_0)U_0^{-1}(\vec{\tau}_0) = e^{iA\tau_0}\cdot U_1(\vec{\tau}_1)U_0^{-1}(\vec{\tau}_1)e^{-iA\tau_0} - \xi_1 \frac{\partial U_0}{\partial\tau_1} U_0^{-1} + i\xi_0 A + \\ + i\int_0^{\tau_0} d\lambda\widetilde{\Gamma}(\lambda) \tag{1.19}$$

A simple manipulation yields

$$e^{-iA\tau_0} \frac{1}{\xi_1} U_1(\vec{\tau}_0)U_0^{-1}(\vec{\tau}_0)e^{iA\tau_0} = \frac{1}{\xi_1} U_1(\vec{\tau}_1)U_0^{-1}(\vec{\tau}_1) - i\Delta_1 + i\frac{\xi_0}{\xi_1} A + \\ + \frac{i}{\xi_1}\int_0^{\tau_0} d\lambda\widetilde{\Gamma}(-\lambda) \tag{1.20}$$

where we have introduced

$$\Delta_1 = -i\frac{\partial U_0(\vec{\tau}_1)}{\partial\tau_1} U_0^{-1}(\vec{\tau}_1) \tag{1.21}$$

We obtain

$$\lim_{\tau_0\to\infty} e^{-iA\tau_0} \frac{1}{\xi_1} U_1(\vec{\tau}_0)U_0^{-1}(\vec{\tau}_0)e^{iA\tau_0} = 0 \tag{1.22}$$

providing that the following condition is fulfilled

$$\Delta_1 = \lim_{\tau_0\to\infty}\left\{\frac{\xi_0}{\xi_1} A + \frac{1}{\xi_1}\int_0^{\tau_0} d\lambda\widetilde{\Gamma}(-\lambda)\right\} \tag{1.23}$$

We can make the calculation more explicit by using a complete set of eigenfunctions of the unperturbed operator. We let

$$A|n\rangle = a_n|n\rangle \tag{1.24}$$

We can then obtain matrix representations in terms of the basic

eigenvectors as

$$A = \sum_n a_n |n\rangle \langle n| \tag{1.25}$$

$$\Gamma = \sum_{m,n} \Gamma_{mn} |m\rangle \langle n| \tag{1.26}$$

We have then the following expression for $\widetilde{\Gamma}$

$$\widetilde{\Gamma}(-\lambda) = e^{-iA\lambda} \sum_{m,n} \Gamma_{m,n} |m\rangle \langle n| e^{iA\lambda}$$

$$= \sum_{m,n} \Gamma_{mn}\, e^{-i(a_m - a_n)\lambda} |m\rangle \langle n| \tag{1.27}$$

We readily can split the sum as

$$\widetilde{\Gamma}(-\lambda) = \sum_m \Gamma_{mm} |m\rangle \langle m| + \sum_{m,n}{}' \Gamma_{m,n}\, e^{-i(a_m - a_n)\lambda} |m\rangle \langle n| \tag{1.28}$$

where the prime signifies $m \neq n$. We thus see that (1.23) becomes

$$\Delta_1 = \lim_{\tau_0 \to \infty} \left\{ \frac{\xi_0}{\xi_1} \sum_m a_m |m\rangle \langle m| + \frac{\tau_0}{\xi_1} \sum_m \Gamma_{mm} |m\rangle \langle m| \right\} \tag{1.29}$$

If we now let

$$\xi_1 = \tau_0 \tag{1.30}$$

and

$$\xi_0 = \gamma \tau_0 \tag{1.31}$$

we can write

$$\Delta_1 = \sum_m (\gamma a_m + \Gamma_{mm}) |m\rangle \langle m| \tag{1.32}$$

Furthermore,

$$\Delta_1 \cdot A = \sum_m (\gamma a_m + \Gamma_{mm}) |m\rangle \langle m| \sum_n a_n |n\rangle \langle n|$$

$$= \gamma \sum_m a_m a_m |m\rangle \langle m| + \sum_m \Gamma_{mm} a_m |m\rangle \langle m| \tag{1.33}$$

It is readily shown

$$\mathrm{Tr}(\Delta_1 \cdot A) = \gamma \sum_m a_m a_m = 0 \tag{1.34}$$

and we thus conclude

$$\gamma = 0 \quad , \qquad \xi_0 = 0 \tag{1.35}$$

The final expression for Δ_1 reduces to

$$\Delta_1 = \sum_m \Gamma_{mm} |m\rangle \langle m| \tag{1.36}$$

This is one of the principal results of the paper. We notice that the first-order quantity is written as

$$U_1(\vec{\tau}_0) = e^{iA\tau_0} U_1(\vec{\tau}_1) + i\left[\int_0^{\tau_0} d\lambda \widetilde{\Gamma}(\lambda) - \tau_0 \widetilde{\Delta}_1 \right] U_0(\vec{\tau}_0) \tag{1.37}$$

Since we have

$$[\Delta_1, A] = 0 \tag{1.38}$$

we also conclude that

$$\widetilde{\Delta}_1 = \Delta_1 \tag{1.39}$$

To obtain a final expression for the perturbation result, we notice some useful relations, in particular,

$$\widetilde{\Gamma}(\lambda) = \sum_{m,n} \Gamma_{mn} \, e^{i(a_m - a_n)\lambda} |m\rangle \langle n| \tag{1.40}$$

$$\int_0^{\tau_0} d\lambda \widetilde{\Gamma}(\lambda) = \Delta_1 \tau_0 + \sum_{m,n}{}' \Gamma_{mn} \left(\frac{e^{i(a_m - a_n)\tau_0} - 1}{i(a_m - a_n)} \right) |m\rangle \langle n| \tag{1.41}$$

We can then write

$$\int_0^{\tau_0} d\lambda \widetilde{\Gamma}(\lambda) = \Delta_1 \tau_0 + \widetilde{\mathcal{S}}(\tau_0) \tag{1.42}$$

where we have introduced

$$\widetilde{\mathcal{S}}(\lambda) = \sum_{m,n}{}' \Gamma_{m,n} \left(\frac{e^{i(a_m - a_n)\lambda} - 1}{i(a_m - a_n)} \right) |m\rangle \langle n| \tag{1.43}$$

Thus, the first-order expression is given by

$$U_1(\vec{\tau}_0) = e^{iA\tau_0}\, U_1(\vec{\tau}_1) + i\tilde{\mathscr{S}}(\tau_0)\cdot U_0(\vec{\tau}_0) \tag{1.44}$$

Differentiation yields

$$i\,\frac{\partial U_1(\vec{\tau}_0)}{\partial \tau_1} = i e^{iA\tau_0}\,\frac{\partial U_1(\vec{\tau}_1)}{\partial \tau_1} - i\tilde{\mathscr{S}}(\tau_0)\cdot \Delta_1\cdot U_0(\vec{\tau}_0) \tag{1.45}$$

Multiplication by Δ_1 yields

$$\Delta_1 U_1(\vec{\tau}_0) = e^{iA\tau_0}\,\Delta_1\cdot U_1(\vec{\tau}_1) + i\Delta_1\tilde{\mathscr{S}}(\tau_0)\cdot U_0(\vec{\tau}_0) \tag{1.46}$$

We thus obtain

$$i\,\frac{\partial U_1(\vec{\tau}_0)}{\partial \tau_1} + \Delta_1 U_1(\vec{\tau}_0) = e^{iA\tau_0}\left[i\,\frac{\partial U_1(\vec{\tau}_1)}{\partial \tau_1} + \Delta_1 U_1(\vec{\tau}_1)\right] - i\left[\tilde{\mathscr{S}}(\tau_0),\Delta_1\right]\cdot U_0(\vec{\tau}_0) \tag{1.47}$$

The following calculation now indicates the behavior of the quantity $\mathscr{S}$. We have, in fact,

$$\begin{aligned}
\left[\tilde{\mathscr{S}}(\tau_0),\Delta_1\right] &= \sum_{m,n}{}' \Gamma_{m,n}\left(\frac{e^{i(a_m-a_n)\tau_0}-1}{i(a_m-a_n)}\right)\sum_{m'} \Gamma_{m'm'}\Big[\,|m\rangle\langle n|\,,\,|m'\rangle\langle m'|\,\Big] \\
&= \sum_{m,n}{}'\sum_{m'} \Gamma_{mn}\Gamma_{m'm'}\left(\frac{e^{i(a_m-a_n)\tau_0}-1}{i(a_m-a_n)}\right)\Big[\,|m\rangle\langle m'|\,\delta_{nm'} - |m'\rangle\langle n|\,\delta_{m'm}\Big] \\
&= \sum_{m,n}{}' \Gamma_{mn}\Gamma_{nn}\left(\frac{e^{i(a_m-a_n)}-1}{i(a_m-a_n)}\right)|m\rangle\langle n| - \sum_{m,n}{}' \Gamma_{mn}\Gamma_{mm}\left(\frac{e^{i(a_m-a_n)\tau_0}-1}{i(a_m-a_n)}\right)|m\rangle\langle n| \\
&= \sum_{m,n}{}' \Gamma_{mn}(\Gamma_{nn}-\Gamma_{mm})\left(\frac{e^{i(a_m-a_n)\tau_0}-1}{i(a_m-a_n)}\right)|m\rangle\langle n|
\end{aligned} \tag{1.48}$$

We then conclude

$$\left[\widetilde{\mathcal{S}}(\tau_0),\Delta_1\right] = \sum_{m,n}{}' \Gamma_{mn}(\Gamma_{nn} - \Gamma_{mm})\left(\frac{e^{i(a_m-a_n)\tau_0}-1}{i(a_m-a_n)}\right)|m\rangle\langle n| \qquad (1.49)$$

2. SECOND-ORDER PERTURBATION THEORY

The second-order equation is written

$$i \frac{\partial U_2}{\partial \tau_0} + A \cdot U_2 = -i\dot{\xi}_1 \frac{\partial U_1}{\partial \tau_1} - i\dot{\eta}_1 \frac{\partial U_0}{\partial \tau_1} - i\dot{\eta}_2 \frac{\partial U_0}{\partial \tau_2} - \Gamma\cdot U_1 - \dot{\eta}_0 A \cdot U_0$$

$$= -i \frac{\partial U_1}{\partial \tau_1} + \dot{\eta}_1 \Delta_1 U_0 - i\dot{\eta}_2 \frac{\partial U_0}{\partial \tau_2} - \Gamma\cdot U_1 - \dot{\eta}_0 A \cdot U_0 \quad . \tag{2.1}$$

We have to use the following expression

$$i \frac{\partial U_1}{\partial \tau_1} = ie^{iA\tau_0} \frac{\partial U_1(\vec{\tau}_1)}{\partial \tau_1} - i\left[\int_0^{\tau_0} d\lambda \tilde{\Gamma}(\lambda) - \tau_0 \Delta_1 \right] \Delta_1 U_0 \tag{2.2}$$

which was derived from first-order theory. Multiplication by Γ gives

$$\Gamma\cdot U_1 = \Gamma\cdot e^{iA\tau_0} U_1(\vec{\tau}_1) + i\Gamma \int_0^{\tau_0} d\lambda \tilde{\Gamma}(\lambda) \cdot U_0 - i\tau_0 \Gamma \cdot \Delta_1 \cdot U_0(\tau_0) \tag{2.3}$$

The second-order equation can then be written as

$$\begin{aligned} i \frac{\partial U_2}{\partial \tau_0} + A \cdot U_2 = {} & -ie^{iA\tau_0} \frac{\partial U_1(\vec{\tau}_1)}{\partial \tau_1} + i \int_0^{\tau_0} d\lambda \tilde{\Gamma}(\lambda) \Delta_1 U_0 - i\tau_0 \Delta_1 U_0 + \\ & - \Gamma\cdot e^{iA\tau_0} U_1(\tau_1) - i\Gamma \int_0^{\tau_0} d\lambda \tilde{\Gamma}(\lambda) U_0 + i\tau_0 \Gamma \cdot \Delta_1 \cdot U_0 + \\ & + \dot{\eta}_1 \Delta_1 U_0 - i\dot{\eta}_2 \frac{\partial U_0}{\partial \tau_2} - \dot{\eta}_0 A \cdot U_0 \end{aligned} \tag{2.4}$$

The solution of this equation can be written

$$U_2(\tau_0) = -\tau_0 e^{iA\tau_0} \frac{\partial U_1(\vec{\tau}_1)}{\partial \tau_1} + \int_0^{\tau_0} d\lambda \int_\lambda^{\tau_0} d\lambda' \tilde{\Gamma}(\lambda') \Delta_1 U_0(\tau_0) +$$

$$- \frac{1}{2} \tau_0^2 \Delta_1^2 U_0 + i \int_0^{\tau_0} d\lambda \tilde{\Gamma}(\lambda) e^{iA\tau_0} U_1(\vec{\tau}_1) +$$

$$- \int_0^{\tau_0} d\lambda \tilde{\Gamma}(\lambda) \int_\lambda^{\tau_0} d\lambda' \tilde{\Gamma}(\lambda') \cdot U_0 + \int_0^{\tau_0} d\lambda (\tau_0 - \lambda) \tilde{\Gamma}(\lambda) \Delta_1 U_0(\vec{\tau}_0) +$$

$$- i\eta_1 \Delta_1 U_0 - \eta_2 \frac{\partial U_0}{\partial \tau_2} + i\eta_0 A \cdot U_0 \tag{2.5}$$

We now note some integral relationships, in particular,

$$\int_0^{\tau_0} d\lambda \int_\lambda^{\tau_0} d\lambda' \tilde{\Gamma}(\lambda') = \int_0^{\tau_0} d\lambda' \tilde{\Gamma}(\lambda') \int_0^{\lambda'} d\lambda = \int_0^{\tau_0} d\lambda \ \lambda \tilde{\Gamma}(\lambda) \tag{2.6}$$

$$- \int_0^{\tau_0} d\lambda \int_\lambda^{\tau_0} d\lambda' \tilde{\Gamma}(\lambda') - \int_0^{\tau_0} d\lambda (\tau_0 - \lambda) \tilde{\Gamma}(\lambda) = -\tau_0 \int_0^{\tau_0} d\lambda \tilde{\Gamma}(\lambda) \tag{2.7}$$

We can therefore re-express the second-order quantity as

$$U_2(\tau_0) = -\tau_0 e^{iA\tau_0} \frac{\partial U_1(\vec{\tau}_1)}{\partial \tau_1} + \tau_0 \int_0^{\tau_0} d\lambda \tilde{\Gamma}(\lambda) \Delta_1 U_0(\tau_0) - \frac{1}{2} \tau_0^2 \Delta_1^2 U_0 +$$

$$+ i \int_0^{\tau_0} d\lambda \tilde{\Gamma}(\lambda) e^{iA\tau_0} U_1(\vec{\tau}_1) - \int_0^{\tau_0} d\lambda \tilde{\Gamma}(\lambda) \int_\lambda^{\tau_0} d\lambda' \tilde{\Gamma}(\lambda') \cdot U_0 +$$

$$- i\eta_1 \Delta_1 U_0 - \eta_2 \frac{\partial U_0}{\partial \tau_2} + i\eta_0 A \cdot U_0 \tag{2.8}$$

We also note the following decomposition of the double integral

$$\int_0^{\tau_0} d\lambda\tilde{\Gamma}(\lambda)\int_\lambda^{\tau_0} d\lambda'\tilde{\Gamma}(\lambda') = \frac{1}{2}\int_0^{\tau_0} d\lambda\tilde{\Gamma}(\lambda)\int_0^{\tau_0} d\lambda'\tilde{\Gamma}(\lambda') +$$

$$+\frac{1}{2}\int_0^{\tau_0} d\lambda\int_0^{\lambda} d\lambda'\left[\tilde{\Gamma}(\lambda'),\tilde{\Gamma}(\lambda)\right]$$

$$= \frac{1}{2}\left(\Delta_1\tau_0+\tilde{\mathcal{S}}(\tau_0)\right)^2 +$$

$$+\frac{1}{2}\int_0^{\tau_0} d\lambda\int_0^{\lambda} d\lambda'\left[\tilde{\Gamma}(\lambda'),\tilde{\Gamma}(\lambda)\right] \qquad (2.9)$$

We can then write the second-order equation as

$$U_2(\tau_0) = -\eta_2\frac{\partial U_0}{\partial\tau_2} + i\eta_0 A\cdot U_0 - i\eta_1\Delta_1 U_0 + \tau_0(\Delta_1\tau_0+\tilde{\mathcal{S}})\Delta_1 U_0 - \frac{1}{2}\tau_0^2\Delta_1^2 U_0 +$$

$$-\tau_0 e^{iA\tau_0}\frac{\partial U_1(\vec{\tau}_1)}{\partial\tau_1} + i(\Delta_1\tau_0+\tilde{\mathcal{S}})e^{iA\tau_0}U_1(\vec{\tau}_1) +$$

$$-\frac{1}{2}\left(\Delta_1^2\tau_0^2+\tau_0\Delta_1\tilde{\mathcal{S}}+\tau_0\tilde{\mathcal{S}}\Delta_1+\tilde{\mathcal{S}}\tilde{\mathcal{S}}\right)U_0 - \frac{1}{2}\int_0^{\tau_0} d\lambda\int_0^{\lambda} d\lambda'\left[\tilde{\Gamma}(\lambda'),\tilde{\Gamma}(\lambda)\right] \quad (2.10)$$

After some simplification, we obtain

$$U_2(\vec{\tau}_0) = -\eta_2\frac{\partial U_0}{\partial\tau_2} + \eta_0 A\cdot U_0 - i\eta_1\Delta_1 U_0 - \frac{1}{2}\tau_0\left[\Delta_1,\tilde{\mathcal{S}}\right]U_0 +$$

$$-\tau_0 e^{iA\tau_0}\left[\frac{\partial U_1(\vec{\tau}_1)}{\partial\tau_1} - i\Delta_1 U_1(\vec{\tau}_1)\right] + i\tilde{\mathcal{S}}e^{iA\tau_0}U_1(\vec{\tau}_1) +$$

$$-\frac{1}{2}\tilde{\mathcal{S}}\tilde{\mathcal{S}}U_0 - \frac{1}{2}\int_0^{\tau_0} d\lambda\int_0^{\lambda} d\lambda'\left[\tilde{\Gamma}(\lambda'),\tilde{\Gamma}(\lambda)\right]U_0 \qquad (2.11)$$

As in first-order theory, in order to study the secular behavior we multiply by the inverse of the zero-order operator to obtain

$$\frac{1}{\eta_2} U_2(\tau_0)U_0^{-1}(\vec{\tau}_0) = - \frac{\partial U_0}{\partial \tau_2} U_0^{-1} + \frac{\eta_0}{\eta_2} A - i \frac{\eta_1}{\eta_2} \Delta_1 - \frac{1}{2} \frac{\tau_0}{\eta_2}\left[\Delta_1, \tilde{\mathcal{S}}\right] +$$

$$- \frac{\tau_0}{\eta_2} e^{iA\tau_0}\left[\frac{\partial U_1(\vec{\tau}_1)}{\partial \tau_1} - i\Delta_1 U_1(\vec{\tau}_1)\right]U_0^{-1}(\vec{\tau}_1)e^{-iA\tau_0} +$$

$$+ \frac{i}{\eta_2} \tilde{\mathcal{S}} e^{iA\tau_0} U_1(\vec{\tau}_1)U_0^{-1}(\vec{\tau}_1)e^{-iA\tau_0} - \frac{1}{2} \frac{1}{\eta_2} \tilde{\mathcal{S}} \tilde{\mathcal{S}} +$$

$$- \frac{1}{2} \frac{1}{\eta_2} \int_0^{\tau_0} d\lambda \int_0^{\lambda} d\lambda' \left[\tilde{\Gamma}(\lambda'), \tilde{\Gamma}(\lambda)\right] \tag{2.12}$$

We can express the manipulations needed in the following form

$$e^{-i\Delta_1\tau_1} e^{-iA\tau_0} \frac{1}{\eta_2} U_2(\vec{\tau}_0) U_0^{-1}(\vec{\tau}_0)e^{iA\tau_0} e^{i\Delta_1\tau_1} =$$

$$= -i\Delta_2 + \frac{\eta_0}{\eta_2} A - i \frac{\eta_1}{\eta_2} \Delta_1 - \frac{\tau_0}{\eta_2} e^{-i\Delta_1\tau_1}\left[\left(\frac{\partial U_1(\vec{\tau}_1)}{\partial \tau_1} - i\Delta_1 U_1(\vec{\tau}_1)\right)U_0^{-1}(\vec{\tau}_1)+\right.$$

$$\left. + \frac{1}{2}[\Delta_1, \tilde{\mathcal{S}}]\right]e^{i\Delta_1\tau_1} + i \frac{1}{\eta_2} e^{-i\Delta_1\tau_1}\tilde{\mathcal{S}}\cdot U_1(\vec{\tau}_1)U_0^{-1}(\vec{\tau}_1)e^{i\Delta_1\tau_1} +$$

$$- \frac{1}{2} \frac{1}{\eta_2} e^{-i\Delta_1\tau_1}\tilde{\mathcal{S}} \tilde{\mathcal{S}}\cdot e^{i\Delta_1\tau_1} - \frac{1}{2\eta_2} e^{-i\Delta_1\tau_1} e^{-iA\tau_0} \cdot$$

$$\cdot \int_0^{\tau_0} d\lambda \int_0^{\lambda} d\lambda' \left[\tilde{\Gamma}(\lambda'), \tilde{\Gamma}(\lambda)\right]e^{iA\tau_0} e^{i\Delta_1\tau_1} \tag{2.13}$$

The secular part is then obtained in the following fundamental expression

$$i\Delta_2 = \lim_{\tau_0 \to \infty} \left\{ \frac{\eta_0}{\eta_2} A - i \frac{\eta_1}{\eta_2} \Delta_1 + \right.$$

$$- \frac{\tau_0}{\eta_2} e^{-i\Delta_1\tau_1}\left[\left(\frac{\partial U_1(\vec{\tau}_1)}{\partial \tau_1} - i\Delta_1 U_1(\vec{\tau}_1)\right)U_0^{-1}(\vec{\tau}_1) + \tfrac{1}{2}[\Delta_1, \tilde{\mathcal{S}}]\right]e^{i\Delta_1\tau_1} +$$

$$\left. - \frac{1}{2\eta_2} e^{-i\Delta_1\tau_1} e^{iA\tau_0}\int_0^{\tau_0} d\lambda \int_0^{\lambda} d\lambda' \left[\tilde{\Gamma}(\lambda'), \tilde{\Gamma}(\lambda)\right] e^{iA\tau_0} e^{i\Delta_1\tau_1}\right\} \quad (2.14)$$

We will now perform several operations to simplify this result. We notice

$$\int_0^{\tau_0} d\lambda \int_0^{\lambda} d\lambda' \left[\tilde{\Gamma}(\lambda'), \tilde{\Gamma}(\lambda)\right] = \tau_0\left[\Delta_1, \tilde{\mathcal{S}}\right] + 2\int_0^{\tau_0} d\lambda \left[\tilde{\mathcal{S}}(\lambda), \Delta_1\right] +$$

$$+ \int_0^{\tau_0} d\lambda \left[\tilde{\mathcal{S}}(\lambda), \dot{\tilde{\mathcal{S}}}(\lambda)\right] \quad (2.15)$$

so that we can rewrite the quantity Δ_2 as

$$i\Delta_2 = \lim_{\tau_0 \to \infty} \left\{ \frac{\eta_0}{\eta_2} A - i \frac{\eta_1}{\eta_2} \Delta_1 + \right.$$

$$- \frac{\tau_0}{\eta_2} e^{-i\Delta_1\tau_1}\left[\left(\frac{\partial U_1(\vec{\tau}_1)}{\partial \tau_1} - i\Delta_1 U_1(\vec{\tau}_1)\right)U_0^{-1}(\vec{\tau}_1) + \tfrac{1}{2}[\Delta_1, \mathcal{S}]\right]e^{i\Delta_1\tau_1} +$$

$$- \frac{1}{2}\frac{\tau_0}{\eta_2} e^{-i\Delta_1\tau_1}[\Delta_1, \mathcal{S}]e^{i\Delta_1\tau_1} - \frac{1}{\eta_2} e^{-i\Delta_1\tau_1} e^{-iA\tau_0} \cdot$$

$$\left. \cdot \int_0^{\tau_0} d\lambda \left[\tilde{\mathcal{S}}(\lambda), \Delta_1\right] e^{iA\tau_0} e^{i\Delta_1\tau_1} - \frac{1}{2\eta_2}\int_0^{\tau_0} d\lambda \left[\tilde{\mathcal{S}}(\lambda), \dot{\tilde{\mathcal{S}}}(\lambda)\right]\right\} \quad (2.16)$$

After some manipulations, we obtain

$$i\Delta_2 = \lim_{\tau_0 \to \infty} \left\{ \frac{\eta_0}{\eta_2} A - i \frac{\eta_1}{\eta_2} \Delta_1 + \right.$$

$$- \frac{\tau_0}{\eta_2} e^{-i\Delta_1\tau_1} \left[\left(\frac{\partial U_1(\vec{\tau}_1)}{\partial \tau_1} - i\Delta_1 U_1(\vec{\tau}_1) \right) U_0^{-1}(\vec{\tau}_1) + [\Delta_1, \mathcal{S}] \right] e^{i\Delta_1\tau_1} +$$

$$- \frac{1}{\eta_2} e^{-i\Delta_1\tau_1} e^{-iA\tau_0} \int_0^{\tau_0} d\lambda \left[\tilde{\mathcal{S}}(\lambda), \Delta_1 \right] e^{iA\tau_0} e^{i\Delta_1\tau_1} +$$

$$\left. - \frac{1}{2\eta_2} e^{-i\Delta_1\tau_1} \int_0^{\tau_0} d\lambda \left[\tilde{\mathcal{S}}(\lambda), \dot{\tilde{\mathcal{S}}}(\lambda) \right] e^{iA\tau_0} e^{i\Delta_1\tau_1} \right\} \tag{2.17}$$

It is useful to recall the following relationship

$$\left[\tilde{\mathcal{S}}(\lambda), \Delta_1 \right] = \sum_{m,n}{}' \Gamma_{mn} (\Gamma_{mn} - \Gamma_{mm}) \left(\frac{e^{i(a_m - a_n)\lambda} - 1}{i(a_m - a_n)} \right) |m\rangle \langle n| \tag{2.18}$$

as well as the integral relation

$$\int_0^{\tau_0} d\lambda \left[\tilde{\mathcal{S}}(\lambda), \Delta_1 \right] = \sum_{m,n}{}' \Gamma_{m,n} (\Gamma_{nn} - \Gamma_{mm}) \left(\frac{e^{i(a_m - a_n)\tau_0} - 1}{i(a_m - a_n)^2} - \frac{\tau_0}{i(a_m - a_n)} \right) |m\rangle \langle n| \tag{2.19}$$

The matrix representations for $\tilde{\mathcal{S}}$ and its derivative are very useful

$$\tilde{\mathcal{S}}(\lambda) = \sum_{m,n}{}' \Gamma_{m,n} \left(\frac{e^{i(a_m - a_n)\lambda} - 1}{i(a_m - a_n)} \right) |m\rangle \langle n| \tag{2.20}$$

$$\dot{\tilde{\mathcal{S}}}(\lambda) = \sum_{m,n}{}' \Gamma_{mn}\, e^{i(a_m - a_n)\lambda} |m\rangle \langle n| \tag{2.21}$$

The commutator of $\mathcal{S}$ with $\tilde{\mathcal{S}}$ can therefore be written

$$\left[\tilde{\mathcal{S}}(\lambda),\dot{\tilde{\mathcal{S}}}(\lambda)\right] = \sum_{m,n}{}' \sum_{m',n'}{}' \Gamma_{mn}\Gamma_{m'n'}\left(\frac{e^{i(a_m-a_n)\lambda}-1}{i(a_m-a_n)}\right)\left(e^{i(a_{m'}-a_{n'})\lambda}\right) \times$$

$$\times \left[|m\rangle\langle n|, |m'\rangle\langle n'|\right]$$

$$= \sum_{\substack{m,n\\ m',n'}}{}' \Gamma_{mn}\Gamma_{m'n'}\left(\frac{e^{i(a_m-a_n)\lambda}-1}{i(a_m-a_n)}\right)e^{i(a_{m'}-a_{n'})\lambda}\Big[|m\rangle\langle n'|\delta_{m'n} +$$

$$- |m'\rangle\langle n|\delta_{n'm}\Big]$$

$$= \sum_{m,n}{}' \Gamma_{mn} \sum_{\substack{n'\\ n\neq n'}} \Gamma_{nn'}\left(\frac{e^{i(a_m-a_n)\lambda}-1}{i(a_m-a_n)}\right)e^{i(a_n-a_{n'})\lambda}|m\rangle\langle n'| +$$

$$- \sum_{m,n}{}' \Gamma_{mn} \sum_{\substack{m'\\ m'\neq m}} \Gamma_{m'm}\left(\frac{e^{i(a_m-a_n)\lambda}-1}{i(a_m-a_n)}\right)e^{i(a_{m'}-a_m)\lambda}|m'\rangle\langle n| \tag{2.22}$$

Some manipulations simplify this expression to

$$\left[\tilde{\mathcal{S}}(\lambda),\dot{\tilde{\mathcal{S}}}(\lambda)\right] = \sum_{m,n}{}' \sum_{\substack{m'\\ n\neq m'}} \Gamma_{mn}\Gamma_{nm'}\left(\frac{e^{i(a_m-a_n)\lambda}-1}{i(a_m-a_n)}\right)e^{i(a_n-a_{m'})\lambda}|m\rangle\langle m'| +$$

$$- \sum_{m,n}{}' \sum_{\substack{m'\\ m'\neq n}} \Gamma_{nm}\Gamma_{m'n}\left(\frac{e^{i(a_n-a_m)\lambda}-1}{i(a_n-a_m)}\right)e^{i(a_{m'}-a_n)\lambda}|m'\rangle\langle m| \tag{2.23}$$

We can combine the terms in the double sum in the following way

$$\left[\tilde{\mathcal{S}}(\lambda),\dot{\tilde{\mathcal{S}}}(\lambda)\right] = \sum_{m,n}{}' \sum_{\substack{m' \\ n\neq m'}} \left\{\Gamma_{mn}\Gamma_{nm'}\left(\frac{e^{i(a_m-a_{m'})\lambda}-e^{i(a_n-a_{m'})\lambda}}{i(a_m-a_n)}\right)|m\rangle\langle m'| + \right.$$

$$\left. - \Gamma_{nm}\Gamma_{m'n}\left(\frac{e^{i(a_{m'}-a_m)\lambda}-e^{i(a_{m'}-a_n)\lambda}}{i(a_n-a_m)}\right)|m'\rangle\langle m|\right\} \qquad (2.24)$$

The secular part of the integral can now be extracted as

$$\text{Sec}\int_0^{\tau_0} d\lambda\left[\tilde{\mathcal{S}}(\lambda),\dot{\tilde{\mathcal{S}}}(\lambda)\right] = \tau_0 \sum_{m,n}{}' \left\{\Gamma_{mn}\Gamma_{nm}\frac{|m\rangle\langle m|}{i(a_m-a_n)} + \Gamma_{nm}\Gamma_{mn}\frac{|m\rangle\langle m|}{i(a_m-a_n)}\right\} \qquad (2.25)$$

This expression can be rewritten as

$$\text{Sec}\int_0^{\tau_0} d\lambda\left[\tilde{\mathcal{S}}(\lambda),\dot{\tilde{\mathcal{S}}}(\lambda)\right] = -2i\tau_0 \sum_{m,n}{}' \Gamma_{mn}\Gamma_{nm}\frac{|m\rangle\langle m|}{(a_m-a_n)} \qquad (2.26)$$

We are now ready to obtain a simple expression for the quantity Δ_2 . Substituting (2.26), we obtain

$$i\Delta_2 = \lim_{\tau_0\to\infty}\left\{\frac{\eta_0}{\eta_2}A - i\frac{\eta_1}{\eta_2}\Delta_1 + \right.$$

$$- \frac{\tau_0}{\eta_2} e^{-i\Delta_1\tau_1}\left[\left(\frac{\partial U_1(\vec{\tau}_1)}{\partial\tau_1} - i\Delta_1 U_1(\vec{\tau}_1)\right)U_0^{-1}(\vec{\tau}_1) + [\Delta_1,\mathcal{S}]\right]e^{i\Delta_1\tau_1} +$$

$$+ \frac{\tau_0}{\eta_2} e^{-i\Delta_1\tau_1}\sum_{m,n}{}'\Gamma_{mn}(\Gamma_{nn}-\Gamma_{mm})\left(\frac{e^{-i(a_m-a_n)\tau_0}}{i(a_m-a_n)}\right)|m\rangle\langle n| e^{i\Delta_1\tau_1} +$$

$$\left. + i\frac{\tau_0}{\eta_2} e^{-i\Delta_1\tau_1}\sum_{m,n}{}'\Gamma_{mn}\Gamma_{nm}\frac{|m\rangle\langle m|}{(a_m-a_n)} e^{i\Delta_1\tau_1}\right\} \qquad (2.27)$$

We notice that this expression can be further manipulated as

$$\Delta_2 = \lim_{\tau_0 \to \infty} \left\{ -i \frac{\eta_0}{\eta_2} A - \frac{\eta_1}{\eta_2} \Delta_1 + \right.$$

$$+ \frac{\tau_0}{\eta_2} e^{-i\Delta_1\tau_1} \left[\left(i\frac{\partial U_1(\dot{\tau}_1)}{\partial \tau_1} + \Delta_1 U_1(\dot{\tau}_1) \right) U_0^{-1}(\dot{\tau}_1) + i[\Delta_1, \mathscr{S}] \right] e^{i\Delta_1\tau_1} +$$

$$- i \frac{\tau_0}{\eta_2} e^{-i\Delta_1\tau_1} \sum_{m,n}{}' \Gamma_{mn}(\Gamma_{nn}-\Gamma_{mm}) \left(\frac{e^{-i(a_m-a_n)\tau_0}}{i(a_m-a_n)} \right) |m\rangle \langle n| \; e^{i\Delta_1\tau_1} +$$

$$\left. + \frac{\tau_0}{\eta_2} e^{-i\Delta_1\tau_1} \sum_{m,n}{}' \Gamma_{mn}\Gamma_{nm} \frac{|m\rangle \langle m|}{(a_m-a_n)} \; e^{i\Delta_1\tau_1} \right\} \tag{2.28}$$

It is important to express the commutation rules of Δ_1 with $\mathscr{S}$. We obtain the following relations

$$\left[\Delta_1, \tilde{\mathscr{S}} \right] = -\sum_{m,n}{}' \Gamma_{mn}(\Gamma_{nn}-\Gamma_{mm}) \left(\frac{e^{i(a_m-a_n)\lambda} - 1}{i(a_m-a_n)} \right) |m\rangle \langle n| \tag{2.29}$$

$$\left[\Delta_1, \mathscr{S} \right] = e^{-iA\lambda} \left[\Delta_1, \tilde{\mathscr{S}} \right] e^{iA\lambda}$$

$$= -\sum_{m,n}{}' \Gamma_{mn}(\Gamma_{nn}-\Gamma_{mm}) \left(\frac{1 - e^{-i(a_m-a_n)\lambda}}{i(a_m-a_n)} \right) |m\rangle \langle n| \tag{2.30}$$

We can thus write

$$\left[\Delta_1, \mathscr{S} \right] = \sum_{m,n}{}' \Gamma_{mn}(\Gamma_{nn}-\Gamma_{mm}) \left(\frac{e^{-i(a_m-a_n)\lambda} - 1}{i(a_m-a_n)} \right) |m\rangle \langle n| \tag{2.31}$$

A simpler expression for Δ_2 that results is given by

$$\Delta_2 = \lim_{\tau_0 \to \infty} \left\{ -i \frac{\eta_0}{\eta_2} A - \frac{\eta_1}{\eta_2} \Delta_1 + \right.$$

$$+ \frac{\tau_0}{\eta_2} e^{-i\Delta_1 \tau_1} \left[\left(i \frac{\partial U_1(\vec{\tau}_1)}{\partial \tau_1} + \Delta_1 U_1(\vec{\tau}_1) \right) U_0^{-1}(\vec{\tau}_1) + \right.$$

$$\left. - i \sum_{m,n}{}' \Gamma_{mn} \left(\frac{\Gamma_{nn} - \Gamma_{mm}}{i(a_m - a_n)} \right) |m\rangle \langle n| \right] e^{i\Delta_1 \tau_1} +$$

$$\left. + \frac{\tau_0}{\eta_2} e^{-i\Delta_1 \tau_1} \sum_{m,n}{}' \Gamma_{mn} \Gamma_{nm} \frac{|m\rangle \langle m|}{(a_m - a_n)} e^{i\Delta_1 \tau_1} \right\} \tag{2.32}$$

If we now consider the following two operators

$$O_1 = \sum_{m,n}{}' \frac{\Gamma_{mn}(\Gamma_{nn} - \Gamma_{mm})}{i(a_m - a_n)} |m\rangle \langle n| \tag{2.33}$$

$$O_2 = \sum_{m,n}{}' \Gamma_{mn} \Gamma_{nm} \frac{|m\rangle \langle m|}{(a_m - a_n)} \tag{2.34}$$

we obtain the following relation

$$[\Delta_1, O_2] = \sum_{m'} \Gamma_{m'm'} \sum_{m,n}{}' \frac{\Gamma_{mn} \Gamma_{nm}}{(a_m - a_n)} [|m'\rangle \langle m'|, |m\rangle \langle m|]$$

$$= \sum_{m,n}{}' \sum_{m'} \Gamma_{m'm'} \left(\frac{\Gamma_{mn} \Gamma_{nm}}{(a_m - a_n)} \right) \left[|m'\rangle \langle m| \delta_{m'm} - |m\rangle \langle m'| \delta_{mm'} \right] = 0 \tag{2.35}$$

which can be summarized as

$$[\Delta_1, O_2] = 0 \tag{2.36}$$

Similarly, we obtain

$$\left[\Delta_1, O_1\right] = \sum_{m'} \Gamma_{m'm'} \sum_{m,n}{}' \frac{\Gamma_{mn}(\Gamma_{nn}-\Gamma_{mm})}{i(a_m-a_n)} \left[|m'\rangle\langle m'|, |m\rangle\langle n|\right]$$

$$= \sum_{m,n}{}' \sum_{m'} \Gamma_{m'm'} \left(\frac{\Gamma_{mn}(\Gamma_{nn}-\Gamma_{mm})}{i(a_m-a_n)}\right) \left[|m'\rangle\langle n| \delta_{m'm} - |m\rangle\langle m'| \delta_{nm'}\right]$$

$$= \sum_{m,n}{}' \frac{\Gamma_{mn}(\Gamma_{nn}-\Gamma_{mm})}{i(a_m-a_n)} \left[\Gamma_{mm}|m\rangle\langle n| - |m\rangle\langle n|\Gamma_{nn}\right] \tag{2.37}$$

so that we have

$$\left[\Delta_1, O_1\right] = -\sum_{m,n}{}' \Gamma_{mn} \frac{(\Gamma_{nn}-\Gamma_{mm})^2}{i(a_m-a_n)} |m\rangle\langle n| \tag{2.38}$$

By writing

$$i \frac{\partial U_1(\vec{\tau}_1)}{\partial \tau_1} + \Delta_1 U_1(\vec{\tau}_1) = +i \sum_{m,n}{}' \Gamma_{mn}\left(\frac{\Gamma_{nn}-\Gamma_{mm}}{i(a_m-a_n)}\right) |m\rangle\langle n| U_0(\vec{\tau}_1) \tag{2.39}$$

we obtain

$$\Delta_2 = \lim_{\tau_2 \to \infty} \left\{-i \frac{\eta_0}{\eta_2} A - \frac{\eta_1}{\eta_2} \Delta_1 + \frac{\dot{\tau}_0}{\eta_2} \sum_{m,n}{}' \Gamma_{mn}\Gamma_{nm} \frac{|m\rangle\langle m|}{(a_m-a_n)}\right\} \tag{2.40}$$

We are now prepared to determine the scales and we introduce them through

$$\eta_2 = \tau_0 \tag{2.41}$$

$$\eta_0 = \gamma_0 \tau_0 \tag{2.42}$$

$$\eta_1 = \gamma_1 \tau_0 \tag{2.43}$$

We can thus write

$$\Delta_2 = -i\gamma_0 A - \gamma_1 \Delta_1 + \sum_{m,n}{}' \Gamma_{mn}\Gamma_{nm} \frac{|m\rangle \langle m|}{(a_m - a_n)} \tag{2.44}$$

This expression is more compactly written as

$$\Delta_2 = -i\gamma_0 A - \gamma_1 \Delta_1 + O_2 \tag{2.45}$$

Multiplication and the tracing operation yields

$$\Delta_2 \cdot A = -i\gamma_0 A \cdot A - \gamma_1 \Delta_1 \cdot A + O_2 \cdot A \tag{2.46}$$

$$\mathrm{Tr}(\Delta_2 \cdot A) = -i\gamma_0 \; \mathrm{Tr}\; A \cdot A + \mathrm{Tr}(O_2 \cdot A) = 0 \tag{2.47}$$

We therefore obtain

$$\gamma_0 = -i \frac{\mathrm{Tr}(O_2 \cdot A)}{\mathrm{Tr}(A \cdot A)} \tag{2.48}$$

Multiplication and tracing with O_2 yields

$$\begin{aligned} O_2 \cdot A &= \sum_{m,n}{}' \Gamma_{mn}\Gamma_{nm} \frac{|m\rangle \langle m|}{a_m - a_n} \sum_{n'} a_{n'} |n'\rangle \langle n'| \\ &= \sum_{m,n}{}' \frac{\Gamma_{mn}\Gamma_{nm}}{(a_m - a_n)} a_m |m\rangle \langle m| \end{aligned} \tag{2.49}$$

$$\mathrm{Tr}(O_2 \cdot A) = \sum_{m,n}' \Gamma_{mn}\Gamma_{nm}\left(\frac{a_m}{a_m - a_n}\right) \tag{2.50}$$

Thus, we can write

$$\gamma_0 = -i\left[\frac{\sum_{m,n}' \Gamma_{mn}\Gamma_{nm}\left(\frac{a_m}{a_m-a_n}\right)}{\sum_m a_m^2}\right] \tag{2.51}$$

Multiplication and tracing with Δ_2 yields

$$\Delta_2 \cdot \Delta_1 = -i\gamma_0 A \cdot \Delta_1 - \gamma_1 \Delta_1 \cdot \Delta_1 + O_2 \cdot \Delta_1 \tag{2.52}$$

$$\mathrm{Tr}(\Delta_2 \cdot \Delta_1) = 0 = -\gamma_1\, \mathrm{Tr}(\Delta_1 \cdot \Delta_1) + \mathrm{Tr}(O_2 \cdot \Delta_1) \tag{2.53}$$

and thus we obtain

$$\gamma_1 = \frac{\mathrm{Tr}(O_2 \cdot \Delta_1)}{\mathrm{Tr}(\Delta_1 \cdot \Delta_2)} \tag{2.54}$$

Multiplication and tracing with Δ_1 yields

$$\begin{aligned}\Delta_1 \cdot \Delta_1 &= \sum_m \Gamma_{mm} \sum_{m'} \Gamma_{m'm'} |m\rangle \langle m|m'\rangle \langle m| \\ &= \sum_m \Gamma_{mm}\Gamma_{mm} |m\rangle\langle m|\end{aligned} \tag{2.55}$$

$$\mathrm{Tr}(\Delta_1 \cdot \Delta_1) = \sum_m \Gamma_{mm}^2 \tag{2.56}$$

Since we also have

$$O_2 \cdot \Delta_1 = \sum_{m,n}{}' \Gamma_{mn}\Gamma_{nm} \frac{|m\rangle\langle m|}{(a_m - a_n)} \sum_{m'} \Gamma_{m'm'} |m'\rangle\langle m'|$$

$$= \sum_{m,n}{}' \Gamma_{mn}\Gamma_{nm}\Gamma_{mm} \frac{|m\rangle\langle m|}{(a_m - a_n)} \tag{2.57}$$

$$\mathrm{Tr}(O_2 \cdot \Delta_1) = \sum_{m,n}{}' \frac{\Gamma_{mn}\Gamma_{nm}\Gamma_{mm}}{(a_m - a_n)} \tag{2.58}$$

we obtain

$$\gamma_1 = \frac{\displaystyle\sum_{m,h}{}' \frac{\Gamma_{mn}\Gamma_{nm}\Gamma_{mm}}{(a_m - a_n)}}{\displaystyle\sum_m \Gamma_{mm}^2} \tag{2.59}$$

We can then write

$$\Delta_2 = O_2 - \frac{\mathrm{Tr}(O_2 \cdot A)}{\mathrm{Tr}(A \cdot A)} A - \frac{\mathrm{Tr}(O_2 \cdot \Delta_1)}{\mathrm{Tr}(\Delta_1 \cdot O_1)} \Delta_1 \tag{2.60}$$

$$O_2 = \sum_{m,n}{}' \Gamma_{mn}\Gamma_{nm} \frac{|m\rangle\langle m|}{(a_m - a_n)} \tag{2.61}$$

This is a rather compact equation for the second-order uniformization. We can now verify a number of the properties of the second-order uniformization. In fact, we have

$$\left[\sum_{m,n}{}' \Gamma_{mn} \frac{|m\rangle \langle n|}{i(a_m-a_n)}, \Delta_1\right] =$$

$$= \sum_{m,n}{}' \frac{\Gamma_{mn}}{i(a_m-a_n)} \sum_{m'} \Gamma_{m'm'} \left[\, |m\rangle\langle n| , |m'\rangle\langle m'| \,\right]$$

$$= \sum_{m,n}{}' \frac{\Gamma_{mn}}{i(a_m-a_n)} \sum_{m'} \Gamma_{m'm'} \left[\, |m\rangle\langle m'|\delta_{m'n} - |m\rangle\langle n|\delta_{m'm} \right]$$

$$= \sum_{m,n}{}' \frac{\Gamma_{mn}}{i(a_m-a_n)} \left[\Gamma_{nn}|m\rangle \langle n| - |m\rangle\langle n|\Gamma_{mm}\right]$$

$$= \sum_{m,n}{}' \frac{\Gamma_{mn}}{i(a_m-a_n)} (\Gamma_{nn} - \Gamma_{mm})|m\rangle \langle n| \tag{2.62}$$

Since the first-order equation can be written as

$$i \frac{\partial U_1(\vec{\tau}_1)}{\partial \tau_1} + \Delta_1 U_1(\vec{\tau}_1) = +i\left[\left(\sum_{m,n}{}' \frac{\Gamma_{mn}}{i(a_m-a_n)} |m\rangle \langle n| , \Delta_1\right]U_0(\vec{\tau}_1) \tag{2.63}$$

we have

$$U_1(\vec{\tau}_1) = e^{i\Delta_1\tau_1}U_1(\vec{\tau}_2) + \int_0^{\tau_1} d\tau_1' e^{i\Delta_1(\tau_1-\tau_1')}\left[\left(\sum_{m,n}{}' \frac{\Gamma_{mn}}{i(a_m-a_n)}|m\rangle \langle n|\right), \Delta_1\right]U_0(\vec{\tau}_1')$$

$$= e^{i\Delta_1\tau_1}U_1(\vec{\tau}_2) - \int_0^{\tau_1} d\lambda e^{i\Delta_1\lambda}\left[\left(\sum_{m,n}{}' \frac{\Gamma_{mn}}{(a_m-a_n)}|m\rangle \langle n|\right), i\Delta_1\right]e^{-i\Delta_1\lambda} U_0(\vec{\tau}_1) \tag{2.64}$$

After some manipulation, we can write the result as

$$U_1(\vec{\tau}_1) = e^{i\Delta_1\tau_1} U_1(\vec{\tau}_2) + e^{i\Delta_1\tau_1} \sum_{m,n}{}' \frac{\Gamma_{m,n}}{(a_m-a_n)} |m\rangle \langle n| e^{-i\Delta_1\tau_1} U_0(\vec{\tau}_1) +$$

$$- \sum_{m,n}{}' \frac{\Gamma_{mn}}{(a_m-a_n)} |m\rangle \langle n| U_0(\vec{\tau}_1) \tag{2.65}$$

If we now introduce the useful notations

$$\Delta_1|m\rangle = \sum_{m'} \Gamma_{m'm'} |m'\rangle \langle m'|m\rangle = \Gamma_{mm}|m\rangle \tag{2.66}$$

$$\Delta_1|m\rangle = \Gamma_{mm}|m\rangle \tag{2.67}$$

we can write

$$U_1(\vec{\tau}_0) = e^{iA\tau_0} e^{i\Delta_1\tau_1} U_1(\vec{\tau}_2) + \sum_{m,n}{}' \left(\frac{\Gamma_{mn}}{a_m-a_n}\right) \left[e^{i(\Gamma_{mm}-\Gamma_{nn})\tau_1} e^{i(a_m-a_n)\tau_0} |m\rangle \langle n| U_0(\vec{\tau}_0) \right.$$

$$- \sum_{m,n}{}' \left(\frac{\Gamma_{mn}}{a_m-a_n}\right) |m\rangle \langle n| U_0(\tau_0) \tag{2.68}$$

We can then give compact results which are

$$U_1(\vec{\tau}_0) = e^{iA\tau_0} e^{i\Delta\tau_1} U_1(\vec{\tau}_2) + \sum_{m,n}{}' \left(\frac{\Gamma_{mn}}{a_m-a_n}\right) \left[e^{i(a_m-a_n)\tau_0} e^{i(\Gamma_{mm}-\Gamma_{nn})\tau_1} - 1 \right] |m\rangle \langle n| U_0(\vec{\tau}_0) \tag{2.69}$$

$$U_0(\vec{\tau}_0) = e^{iA\tau_0} e^{i\Delta_1\tau_1} e^{i\Delta_2\tau_2} U_0(\vec{\tau}_3) \tag{2.70}$$

A slight rearrangement also yields

$$U_1(\vec{\tau}_1) = e^{i\Delta_1\tau_1}\, U_1(\vec{\tau}_2) + \sum_{m,n}{}' \left(\frac{\Gamma_{mn}}{a_m - a_n}\right)\left[e^{i(\Gamma_{mm}-\Gamma_{nn})\tau_1} - 1\right] |m\rangle \langle n| U_0(\vec{\tau}_0) \tag{2.71}$$

We note that since we have

$$\Delta_2 = O_2 - i\gamma_0 A - \gamma_1 \Delta_1 \tag{2.72}$$

with

$$O_2 = \sum_{m,n}{}' \Gamma_{mn}\Gamma_{nm} \frac{|m\rangle \langle m|}{(a_m - a_n)} \tag{2.73}$$

we can also obtain a matrix representation for O_2 . This is given by

$$\begin{aligned} O_2|m\rangle &= \sum_{m',n'}{}' \Gamma_{m'n'}\Gamma_{n'm'} \frac{|m'\rangle \langle m'|}{(a_{m'} - a_{n'})} |m\rangle \\ &= \sum_{\substack{n \\ n\neq m}} \Gamma_{mn}\Gamma_{nm} \frac{|m\rangle}{(a_m - a_n)} \\ &= \sum_{\substack{n \\ n\neq m}} \frac{\Gamma_{mn}\Gamma_{nm}}{(a_m - a_n)} |m\rangle \end{aligned} \tag{2.74}$$

If we then set

$$O_2|m\rangle = O_m|m\rangle \tag{2.75}$$

we find

$$O_m = \sum_{\substack{n \\ n\neq m}} \frac{\Gamma_{mn}\Gamma_{nm}}{(a_m - a_n)} \tag{2.76}$$

We can express Δ_2 then as

$$\Delta_2|m\rangle = (O_m - i\gamma_0 a_m - \gamma_1 \Gamma_{mm})|m\rangle \tag{2.77}$$

which can be summarized as

$$\Delta_2|m\rangle = \delta_m|m\rangle \tag{2.78}$$

$$\delta_m = O_m - i\gamma_0 a_m - \gamma_1\Gamma_{mm} \tag{2.79}$$

We can finally rearrange the first-order result which can be given as

$$\frac{\partial U_1(\vec{\tau}_0)}{\partial\tau_1} + \Delta_1 U_1(\vec{\tau}_0) = e^{iA\tau_0}\left[i\,\frac{\partial U_1(\vec{\tau}_1)}{\partial\tau_1} + \Delta_1 U_1(\vec{\tau}_1)\right] - i\left[\widetilde{\mathscr{S}}(\tau_0),\Delta_1\right]\cdot U_0(\vec{\tau}_0)$$

$$= ie^{iA\tau_0}\left[\left(\sum_{m,n}{}' \frac{\Gamma_{mn}}{i(a_m-a_n)}|m\rangle\langle n|\right),\Delta_1\right]e^{iA\tau_0}\,U_0(\vec{\tau}_0) +$$

$$- i\sum_{m,n}{}'\Gamma_{mn}(\Gamma_{nn}-\Gamma_{mm})\left(\frac{e^{i(a_m-a_n)\tau_0}-1}{i(a_m-a_n)}\right)|m\rangle\langle n|U_0(\vec{\tau}_0)$$

$$= i\sum_{m,n}{}' \frac{\Gamma_{mn}}{i(a_m-a_n)}\,(\Gamma_{nn}-\Gamma_{mm})e^{i(a_m-a_n)\tau_0}|m\rangle\langle n|U_0(\vec{\tau}_0) +$$

$$- i\sum_{m,n}{}'\Gamma_{mn}(\Gamma_{nn}-\Gamma_{mm})\left(\frac{e^{i(a_m-a_n)\tau_0}-1}{i(a_m-a_n)}\right)|m\rangle\langle n|U_0(\vec{\tau}_0)$$

$$= i\sum_{m,n}{}'\Gamma_{mn}(\Gamma_{nn}-\Gamma_{mm})\,\frac{|m\rangle\langle n|}{i(a_m-a_n)}\,U_0(\vec{\tau}_0)$$

$$= i\left[\left(\sum_{m,n}{}' \frac{\Gamma_{mn}}{i(a_m-a_n)}\,|m\rangle\langle n|\right),\Delta_1\right]\cdot U_0(\vec{\tau}_0) \tag{2.80}$$

which can be written compactly as

$$i\,\frac{\partial U_1(\vec{\tau}_0)}{\partial\tau_1} + \Delta_1 U_1(\vec{\tau}_0) = i\left[\left(\sum_{m,n}{}' \frac{\Gamma_{mn}}{i(a_m-a_n)}\,|m\rangle\langle n|\right),\Delta_1\right]U_0(\vec{\tau}_0) \tag{2.81}$$

Utilizing Eq. (2.32), we now notice that alternative expressions for Δ_2 can be given

$$\Delta_2 = \lim_{\tau_0 \to \infty} \left\{ -i \frac{\eta_0}{\eta_2} A - \frac{\eta_1}{\eta_2} \Delta_1 + \right.$$

$$+ \frac{\tau_0}{\eta_2} e^{i\Delta_1 \tau_1} \left[\left(i \frac{\partial U_1(\vec{\tau}_1)}{\partial \tau_1} + \Delta_1 U_1(\vec{\tau}_1) \right) U_0^{-1}(\vec{\tau}_1) + \right.$$

$$\left. \left. -i \sum_{m,n}' \Gamma_{mn} \frac{\Gamma_{nn} - \Gamma_{mm}}{i(a_m - a_n)} |m\rangle \langle n| \right] e^{i\Delta_1 \tau_1} + \frac{\tau_0}{\eta_2} \sum_{m,n}' \Gamma_{mn} \Gamma_{nm} \frac{|m\rangle \langle m|}{(a_m - a_n)} \right\}$$

$$= \lim_{\tau_0 \to \infty} \left\{ -i \frac{\eta_0}{\eta_2} A - \frac{\eta_1}{\eta_2} \Delta_1 + \frac{\tau_0}{\eta_2} \left[i \frac{\partial}{\partial \tau_1} \left(e^{i\Delta_1 \tau_1} U_1(\vec{\tau}_1) \right) U_0^{-1}(\vec{\tau}_2) + \right. \right.$$

$$\left. - i \sum_{m,n}' \frac{\Gamma_{mn}}{i(a_m - a_n)} \left[|m\rangle \langle n|, \Delta_1 \right] e^{-i(\Gamma_{mm} - \Gamma_{nn})\tau_1} \right] +$$

$$\left. + \frac{\tau_0}{\eta_2} \sum_{m,n}' \Gamma_{mn} \Gamma_{nm} \frac{|m\rangle \langle m|}{(a_m - a_n)} \right\} \tag{2.82}$$

We see thus that the second-order theory can be made fully explicit. Operators that uniformize the expansion are computed explicitly and the major assumption, we repeat, has been the linearity of the perturbation expansion in first order and of the quadratic behavior in the second order.

3. DEGENERATE PERTURBATION THEORY

We now consider degenerate perturbation theory by writing the Schrodinger equation as

$$i \frac{\partial V}{\partial \tau} + (A+\epsilon B)V = 0 \qquad (3.1)$$

If we consider the new operator

$$V = e^{iA\tau} . U \qquad (3.2)$$

we thus obtain an interaction representation

$$i \frac{\partial U}{\partial \tau} = -\epsilon \tilde{B} U \qquad (3.3)$$

with the operator

$$\tilde{B} = e^{-iA\tau} . B . e^{iA\tau} \qquad (3.4)$$

We now introduce matrix representations as follows

$$A|m,\alpha\rangle = a_m |m,\alpha\rangle \qquad (3.5)$$

The symbol α denotes the degenerate eigenstates associated with each eigenvalue a_n .

$$B = \sum_{\substack{mn \\ \alpha\beta}} B_{m\alpha,n\beta} |m\alpha\rangle \langle n,\beta| \qquad (3.6)$$

$$\tilde{B} = \sum_{\substack{m,n \\ \alpha,\beta}} B_{m\alpha,n\beta}\, e^{-i(a_m-a_n)\tau} |m\alpha\rangle \langle n\beta| \qquad (3.7)$$

The relevant time-dependent operator given by Eq. (3.7) can be rewritten as

$$\tilde{B}(\tau) = \sum_{\substack{m \\ \alpha\beta}} B_{m\alpha,m\beta} |m\alpha\rangle \langle m\beta| + \sum_{\substack{m\neq n \\ \alpha\beta}} B_{m\alpha,n\beta}\, e^{-i(a_m-a_n)\tau} |m\alpha\rangle \langle n\beta| \quad (3.8)$$

We now consider a linear time scale extension which is given by

$$i \frac{\partial U_0}{\partial \tau_0} = 0 \quad , \qquad U_0(\vec{\tau}_0) = U_0(\vec{\tau}_1) \tag{3.9}$$

The zero in first-order equations yields

$$i \frac{\partial U_0}{\partial \tau_1} + i \frac{\partial U_1}{\partial \tau_0} = -\tilde{B}U_0 \tag{3.10}$$

or

$$i \frac{\partial U_1}{\partial \tau_0} = -i \frac{\partial U_0}{\partial \tau_1} - \tilde{B}U_0 \tag{3.11}$$

We can integrate to yield

$$iU_1(\vec{\tau}_0) = iU_1(\vec{\tau}_1) - i\tau_0 \frac{\partial U_0}{\partial \tau_1} - \int_0^{\tau_0} d\lambda \tilde{B}(\lambda) U_0(\vec{\tau}_1) \tag{3.12}$$

To investigate the secularity, we multiply by the inverse of the zero-order operator obtaining

$$U_1(\vec{\tau}_0)U_0^{-1}(\vec{\tau}_0) = U_1(\vec{\tau}_1)U_0^{-1}(\vec{\tau}_1) - \tau_0 \frac{\partial U_0(\vec{\tau}_1)}{\partial \tau_1} U_0^{-1}(\vec{\tau}_1) + i \int_0^{\tau_0} d\lambda \tilde{B}(\lambda) \tag{3.13}$$

We now define

$$\Delta_1 = -i \frac{\partial U_0(\vec{\tau}_1)}{\partial \tau_1} U_0^{-1}(\vec{\tau}_1) \tag{3.14}$$

We thus obtain

$$U_1(\vec{\tau}_0)U_0^{-1}(\vec{\tau}_0) = U_1(\vec{\tau}_1)U_0^{-1}(\vec{\tau}_1) - i\tau_0\Delta_1 + i\tau_0 \sum_{\substack{m \\ \alpha\beta}} B_{m\alpha,m\beta}|m\alpha\rangle\langle m\beta| +$$

$$+ i \sum_{\substack{m\neq n \\ \alpha\beta}} B_{m\alpha,n\beta}\left(\frac{e^{-i(a_m-a_n)\tau_0}-1}{-i(a_m-a_n)}\right)|m\alpha\rangle\langle n\beta| \tag{3.15}$$

We let

$$\Delta_1 = \sum_{\substack{m \\ \alpha,\beta}} B_{m\alpha,m\beta} \quad |m\alpha\rangle\langle m\beta| \tag{3.16}$$

so that the first-order operator is given by

$$U_1(\vec{\tau}_0) = U_1(\vec{\tau}_1) - \sum_{\substack{m\neq n \\ \alpha\beta}} B_{m\alpha,n\beta} \frac{e^{-i(a_m-a_n)\tau_0}-1}{(a_m-a_n)} |m\alpha\rangle\langle n\beta|U_0(\vec{\tau}_1) \tag{3.17}$$

We note the commutativity

$$[\Delta_1,A] = \sum_{\substack{m \\ \alpha\beta}} B_{m\alpha,m\beta} [|m\alpha\rangle\langle m\beta|,A] = 0 \tag{3.18}$$

The following matrix representations are used extensively

$$\Delta_1 = \sum_m \Delta_1^m \quad , \qquad \Delta_1^m = \sum_{\alpha\beta} B_{m\alpha,m\beta} |m\alpha\rangle\langle m\beta| \tag{3.19}$$

$$A = \sum_m A^m \quad , \qquad A^m = a_m \sum_\alpha |m\alpha\rangle\langle m\alpha| \tag{3.20}$$

We notice

$$[\Delta_1^m,A^m] = 0 \tag{3.21}$$

We thus have a compact version of (3.17) as

$$U_1(\vec{\tau}_0) = U_1(\vec{\tau}_1) + i\left[\int_0^{\tau_0} d\lambda \tilde{B}(\lambda) - \tau_0\Delta_1\right]U_0(\vec{\tau}_1) \tag{3.22}$$

We now define

$$\mathcal{S}(\tau_0) \equiv \int_0^{\tau_0} d\lambda \tilde{B}(\lambda) - \tau_0\Delta_1 \tag{3.23}$$

The matrix representation is given by

$$\mathcal{S}(\tau_0) = i\sum_{\substack{m\neq n\\ \alpha\beta}} B_{m\alpha,n\beta}\left(\frac{e^{-i(a_m-a_n)\tau_0}-1}{a_m-a_n}\right)|m\alpha\rangle\langle n\beta| \tag{3.24}$$

The first-order operator is then given by

$$U_1(\vec{\tau}_0) = U_1(\vec{\tau}_1) + i\mathcal{S}(\tau_0)U_0(\vec{\tau}_1) \tag{3.25}$$

The time derivative of Eq. (3.23) is

$$\dot{\mathcal{S}}(\tau_0) = \tilde{B}(\tau_0) - \Delta_1 \tag{3.26}$$

We can therefore write

$$\frac{\partial U_1}{\partial \tau_1} = \frac{\partial U_1(\vec{\tau}_1)}{\partial \tau_1} + i\mathcal{S}(\tau_0)\frac{\partial U_0(\vec{\tau}_1)}{\partial \tau_1} \tag{3.27}$$

or

$$\frac{\partial U_1}{\partial \tau_1} = \frac{\partial U_1(\vec{\tau}_1)}{\partial \tau_1} - \mathcal{S}(\tau_0)\Delta_1 U_0(\vec{\tau}_1) \tag{3.28}$$

resorting to the matrix representation

$$\dot{\mathcal{S}}(\tau_0) = i\sum_{\substack{m\neq n\\ \alpha\beta}} B_{m\alpha,n\beta}\, e^{-i(a_m-a_n)\tau_0}|m\alpha\rangle\langle n\beta| \tag{3.29}$$

whenever necessary.

We now consider <u>second-order</u> theory. The basic equation is

$$i\,\frac{\partial U_0}{\partial\tau_2}+i\,\frac{\partial U_1}{\partial\tau_1}+i\,\frac{\partial U_2}{\partial\tau_0}=-\tilde{B}U_1 \qquad (3.30)$$

Using the lower-order results, we obtain

$$\begin{aligned}
i\,\frac{\partial U_2}{\partial\tau_0} &= -i\,\frac{\partial U_0}{\partial\tau_2}-i\,\frac{\partial U_1}{\partial\tau_1}-\tilde{B}U_1\\
&= -i\,\frac{\partial U_0}{\partial\tau_2}-i\,\frac{\partial U_1(\vec{\tau}_1)}{\partial\tau_1}+i\mathcal{S}(\tau_0)\Delta_1U_0(\vec{\tau}_1)-\left(\dot{\mathcal{S}}(\tau_0)+\Delta_1\right)\left(U_1(\vec{\tau}_1)+i\mathcal{S}(\tau_0)U_0(\tau_1)\right)\\
&= -i\,\frac{\partial U_0}{\partial\tau_2}-i\,\frac{\partial U_1(\vec{\tau}_1)}{\partial\tau_1}+i\mathcal{S}(\tau_0)\Delta_1U_0(\vec{\tau}_1)-\dot{\mathcal{S}}(\tau_0)U_0(\vec{\tau}_1)-i\dot{\mathcal{S}}(\tau_0)\mathcal{S}(\tau_0)U_0(\vec{\tau}_1)+\\
&\quad -\Delta_1U_1(\vec{\tau}_1)-i\Delta_1\mathcal{S}(\tau_0)U_0(\vec{\tau}_1)\\
&= -i\,\frac{\partial U_0}{\partial\tau_2}-\left(i\,\frac{\partial U_1(\vec{\tau}_1)}{\partial\tau_1}+\Delta_1U_1(\vec{\tau}_1)\right)+i\left[\mathcal{S}(\tau_0),\Delta_1\right]U_0(\vec{\tau}_1)-\dot{\mathcal{S}}(\tau_0)U_0(\vec{\tau}_1)+\\
&\quad -i\dot{\mathcal{S}}(\tau_0)\mathcal{S}(\tau_0)U_0(\tau_1)
\end{aligned} \qquad (3.31)$$

whose integral can be expressed as

$$\begin{aligned}
iU_2(\tau_0) &= iU_2(\vec{\tau}_1)-\tau_0 i\,\frac{\partial U_0}{\partial\tau_2}-\tau_0\left(i\,\frac{\partial U_1(\vec{\tau}_1)}{\partial\tau_1}+\Delta_1U_1(\vec{\tau}_1)\right)+\\
&\quad +i\int_0^{\tau_0}d\lambda[\mathcal{S}(\lambda),\Delta_1]U_0(\vec{\tau}_1)-\mathcal{S}(\tau_0)U_0(\vec{\tau}_1)+\\
&\quad -i\int_0^{\tau_0}d\lambda\dot{\mathcal{S}}(\lambda)\mathcal{S}(\lambda)U_0(\vec{\tau}_1)
\end{aligned} \qquad (3.32)$$

We shall need some integral representations. In particular,

$$\begin{aligned}
\int_0^{\tau_0}d\lambda\dot{\mathcal{S}}(\lambda)\mathcal{S}(\lambda) &= \frac{1}{2}\int_0^{\tau_0}d\lambda[\dot{\mathcal{S}}(\lambda)\mathcal{S}(\lambda)+\mathcal{S}(\lambda)\dot{\mathcal{S}}(\lambda)]+\\
&\quad \frac{1}{2}\int_0^{\tau_0}d\lambda[\dot{\mathcal{S}}(\lambda),\mathcal{S}(\lambda)]\\
&= \frac{1}{2}\mathcal{S}^2(\tau_0)+\frac{1}{2}\int_0^{\tau_0}d\lambda[\dot{\mathcal{S}}(\lambda),\mathcal{S}(\lambda)]
\end{aligned} \qquad (3.33)$$

$$iU_2(\vec{\tau}_0) = iU_2(\vec{\tau}_1) - \tau_0 i\,\frac{\partial U_0}{\partial \tau_2} - \tau_0\left(i\,\frac{\partial U_1(\vec{\tau}_1)}{\partial \tau_1} + \Delta_1 U_1(\vec{\tau}_1)\right) +$$

$$+ i\int_0^{\tau_0} d\lambda[\mathcal{S}(\lambda),\Delta_1]U_0(\vec{\tau}_1) - \mathcal{S}(\tau_0)U_0(\vec{\tau}_1) +$$

$$- i\,\frac{1}{2}\mathcal{S}^2(\tau_0)U_0(\vec{\tau}_1) - i\,\frac{1}{2}\int_0^{\tau_0} d\lambda[\dot{\mathcal{S}}(\lambda),\mathcal{S}(\lambda)]U_0(\vec{\tau}_1) \quad (3.34)$$

To study the secularity, we use the trick of multiplication by the zero-order operator. We obtain

$$U_2(\vec{\tau}_0)U_0^{-1}(\vec{\tau}_0) = U_2(\vec{\tau}_1)U_0^{-1}(\vec{\tau}_1) - \tau_0\,\frac{\partial U_0(\vec{\tau}_1)}{\partial \tau_2}\,U_0^{-1}(\tau_1) +$$

$$+ i\tau_0\left(i\,\frac{\partial U_1(\vec{\tau}_1)}{\partial \tau_1} + \Delta_1 U_1(\vec{\tau}_1)\right)U_0^{-1}(\vec{\tau}_1) +$$

$$+ \int_0^{\tau_0} d\lambda\left[\mathcal{S}(\lambda),\Delta_1\right] + i\mathcal{S}(\tau_0) +$$

$$- \frac{1}{2}\mathcal{S}^2(\tau_0) - \frac{1}{2}\int_0^{\tau_0} d\lambda\left[\dot{\mathcal{S}}(\lambda),\mathcal{S}(\lambda)\right] \quad (3.35)$$

The commutator of $\mathcal{S}$ with Δ_1 can be computed explicitly. The calculation yields

$$[\mathcal{S}(\tau),\Delta_1] =$$

$$= i\sum_{\substack{m\neq n\\ \alpha\beta}} B_{m\alpha,n\beta}\left(\frac{e^{-i(a_m-a_n)\lambda}-1}{a_m-a_n}\right)\sum_{\substack{m'\\ \mu\nu}} B_{m'\mu,m'\nu}\Big[|m\alpha\rangle\langle n\beta|\,,\,|m'\mu\rangle\langle m'\nu|\Big]$$

$$= i\sum_{\substack{m\neq n\\ \alpha\beta}}\sum_{\substack{m'\\ \mu\nu}} B_{m\alpha,n\beta}B_{m'\mu,m'\nu}\left(\frac{e^{-i(a_m-a_n)\lambda}-1}{a_m-a_n}\right)\Big[|m\alpha\rangle\langle m'\nu|\delta_{m'n}\delta_{\mu\beta} +$$

$$- |m'\mu\rangle\langle n\beta|\delta_{m'm}\delta_{\alpha\nu}\Big] =$$

$$= i \sum_{\substack{m\neq n\\ \alpha\beta}} B_{m\alpha,n\beta}\left(\frac{e^{-i(a_m-a_n)\lambda}-1}{a_m-a_n}\right)\left\{\sum_{\nu} B_{n\beta,n\nu}\,|m\alpha\rangle\,\langle n\nu| \;+\right.$$

$$\left. -\; B_{m\mu,m\alpha}\,|m\nu\rangle\,\langle n\beta|\right\}$$

$$= i \sum_{\substack{m\neq n\\ \alpha\beta\\ \nu}} B_{m\alpha,n\beta}B_{n\beta,n\nu}\left(\frac{e^{-i(a_m-a_n)\lambda}-1}{a_m-a_n}\right)|m\alpha\rangle\,\langle n\nu| \;+$$

$$= i \sum_{\substack{m\neq n\\ \alpha\beta\\ \nu}} B_{n\alpha,m\beta}\left(\frac{e^{-i(a_n-a_m)\lambda}-1}{a_n-a_m}\right)B_{n\nu,n\alpha}\,|n\nu\rangle\,\langle m\beta|$$

$$= i \sum_{\substack{m\neq n\\ \alpha\beta\\ \nu}} B_{m\alpha,n\beta}B_{n\beta,n\nu}\left(\frac{e^{-i(a_m-a_n)\lambda}-1}{a_m-a_n}\right)|m\alpha\rangle\,\langle n\nu| \;+$$

$$+\; i \sum_{\substack{m\neq n\\ \alpha\beta\\ \nu}} B_{n\beta,m\alpha}B_{n\nu,n\beta}\left(\frac{e^{i(a_m-a_n)\lambda}-1}{a_m-a_n}\right)|n\nu\rangle\,\langle m\alpha| \qquad (3.36)$$

The integral of the commutator is also of importance

$$^{0}\, d\lambda[\mathcal{V}(\lambda),\Delta_1] =$$

$$= -\sum_{\substack{m\neq n\\ \alpha\beta\\ \nu}} B_{m\alpha,n\beta}B_{n\beta,n\nu}\left(\frac{e^{-i(a_m-a_n)\tau_0}-1}{(a_m-a_n)^2}\right)|m\alpha\rangle\,\langle n\nu| \;+$$

$$+\sum_{\substack{m\neq n\\ \alpha\beta\\ \nu}} B_{n\nu,n\beta}B_{n\beta,m\alpha}\left(\frac{e^{i(a_m-a_n)\tau_0}-1}{(a_m-a_n)^2}\right)|n\nu\rangle\,\langle m\alpha| \;+$$

$$- i\tau_0 \sum_{\substack{m\neq n\\ \alpha\beta\\ \nu}} B_{m\alpha,n\beta} B_{n\beta,n\nu} \frac{|m\alpha\rangle \langle n\nu|}{(a_m - a_n)} +$$

$$- i\tau_0 \sum_{\substack{m\neq n\\ \alpha\beta\\ \nu}} B_{n\nu,n\beta} B_{n\beta,m\alpha} \frac{|n\nu\rangle \langle m\alpha|}{(a_m - a_n)} \qquad (3.37)$$

The commutator of $\mathcal{S}$ with the time derivative is also computed explicitly. We first obtain

$$[\dot{\mathcal{S}}(\lambda),\mathcal{S}(\lambda)] = -\sum_{\substack{m\neq n\\ \alpha\beta}} \sum_{\substack{p\neq q\\ \mu\nu}} B_{m\alpha,n\beta}\, e^{-i(a_m-a_n)\lambda} B_{p\mu,q\nu} \left(\frac{e^{-i(a_p-a_q)\lambda}-1}{a_p-a_q}\right) \cdot$$

$$\cdot [|m\alpha\rangle \langle n\beta|, |p\mu\rangle \langle q\nu|]$$

$$= -\sum_{\substack{m\neq n\\ \alpha\beta}} \sum_{\substack{p\neq q\\ \mu\nu}} B_{m\alpha,n\beta}\, B_{p\mu,q\nu}\, e^{-i(a_m-a_n)\lambda} \left(\frac{e^{-i(a_p-a_q)\lambda}-1}{q_p-a_q}\right) \cdot$$

$$\cdot \left[|m\alpha\rangle \langle q\nu|\ \delta_{\mu p}\delta_{\beta\mu} - |p\mu\rangle \langle \mu\beta|\delta_{qm}\delta_{\nu\alpha}\right]$$

$$= -\sum_{\substack{m\neq n\\ \alpha\beta}} \sum_{\substack{q\neq n\\ \nu}} B_{m\alpha,n\beta} B_{n\beta,q\nu}\, e^{-i(a_m-a_n)\lambda} \left(\frac{e^{-i(a_m-a_q)\lambda}-1}{a_n-a_q}\right) |m\alpha\rangle\langle q\nu| +$$

$$+ \sum_{\substack{m\neq n\\ \alpha\beta}} \sum_{\substack{p\neq m\\ \mu}} B_{m\alpha,n\beta} B_{p\mu,m\alpha} e^{-i(a_m-a_n)\lambda} \left(\frac{e^{-i(a_p-a_m)\lambda}-1}{a_p-a_m}\right) |p\mu\rangle\langle n\beta|$$

$$= -\sum_{\substack{m\neq n\\ \alpha\beta}} \sum_{\substack{p\neq n\\ \mu}} B_{m\alpha,n\beta} B_{n\beta,p\mu} e^{-i(a_m-a_n)\lambda} \left(\frac{e^{-i(a_n-a_p)\lambda}-1}{a_n-a_p}\right) |m\alpha\rangle\langle p\mu| +$$

$$+\sum_{\substack{m\neq\mu\\ \alpha\beta}}\sum_{\substack{p\neq n\\ \mu}} B_{n\beta,m\alpha}B_{p\mu,n\beta}e^{-i(a_n-a_m)\lambda}\left(\frac{e^{-i(a_p-a_n)\lambda}-1}{a_p-a_n}\right)|p\mu\rangle\langle m\alpha| \tag{3.38}$$

Some manipulations are needed in order to write this expression as

$$[\dot{\mathscr{J}}(\lambda),\mathscr{J}(\lambda)] =$$

$$=\sum_{\substack{m\neq n\\ \alpha\beta}}\sum_{\substack{p\neq n\\ \mu}} B_{m\alpha,n\beta}B_{n\beta,p\mu}\left(\frac{e^{-i(a_m-a_p)\lambda}-e^{-i(a_m-a_n)\lambda}}{a_p-a_n}\right)|m\alpha\rangle\langle p\mu| +$$

$$+\sum_{\substack{m\neq n\\ \alpha\beta}}\sum_{\substack{p\neq n\\ \mu}} B_{p\mu,n\beta}B_{n\beta,m\alpha}\left(\frac{e^{-i(a_p-a_m)\lambda}-e^{-i(a_n-a_m)\lambda}}{a_p-a_n}\right)|p\mu\rangle\langle m\alpha|$$

$$=-\sum_{\substack{m\neq n\\ \alpha\beta}}\sum_{\substack{p\neq n\\ \mu}} B_{m\alpha,n\beta}B_{n\beta,p\mu}\left(\frac{e^{-i(a_m-a_n)\lambda}}{a_p-a_n}\right)|m\alpha\rangle\langle p\mu| +$$

$$-\sum_{\substack{m\neq n\\ \alpha\beta}}\sum_{\substack{p\neq n\\ \mu}} B_{p\mu,n\beta}B_{n\beta,m\alpha}\left(\frac{e^{-i(a_n-a_m)\lambda}}{a_p-a_n}\right)|p\mu\rangle\langle m\alpha| +$$

$$+\sum_{\substack{m\neq n\\ \alpha\beta}}\sum_{\substack{p\neq n\\ p\neq m\\ \mu}} B_{m\alpha,n\beta}B_{n\beta,p\mu}\left(\frac{e^{-i(a_m-a_p)\lambda}}{a_p-a_n}\right)|m\alpha\rangle\langle p\mu| +$$

$$+\sum_{\substack{m\neq n\\ \alpha\beta}}\sum_{\substack{p\neq n\\ p\neq m\\ \mu}} B_{p\mu,n\beta}B_{n\beta,m\alpha}\left(\frac{e^{-i(a_p-a_m)\lambda}}{a_p-a_n}\right)|p\mu\rangle\langle m\alpha| +$$

$$+\sum_{\substack{m\neq n\\ \alpha\beta\mu}} B_{m\alpha,n\beta}B_{n\beta,m\mu}\,\frac{|m\alpha\rangle\langle m\mu|}{a_m-a_n} +$$

$$+\sum_{\substack{m\neq n\\ \alpha\beta\mu}} B_{m\mu,n\beta}\,B_{n\beta,m\alpha}\,\frac{|m\mu\rangle\langle m\alpha|}{a_m-a_n} \tag{3.39}$$

We also will need

$$
\begin{aligned}
d\lambda[\mathcal{S}(\lambda),\mathcal{S}(\lambda)] =& \\
=& -i\sum_{\substack{m\neq n\\ \alpha\beta}}\sum_{\substack{p\neq n\\ \mu}} B_{m\alpha,n\beta}B_{n\beta,p\mu}\left(\frac{e^{-i(a_m-a_n)\tau_0}-1}{(a_m-a_n)(a_p-a_n)}\right)|m\alpha\rangle\langle p\mu| + \\
&+ i\sum_{\substack{m\neq n\\ \alpha\beta}}\sum_{\substack{p\neq n\\ \mu}} B_{p\mu,n\beta}B_{n\beta,m\alpha}\left(\frac{e^{i(a_m-a_n)\tau_0}-1}{(a_m-a_n)(a_p-a_n)}\right)|p\mu\rangle\langle m\alpha| + \\
&+ i\sum_{\substack{m\neq n\\ \alpha\beta}}\sum_{\substack{p\neq n\\ p\neq m\\ \mu}} B_{m\alpha,n\beta}B_{n\beta,p\mu}\left(\frac{e^{-i(a_m-a_p)\tau_0}-1}{(a_m-a_p)(a_p-a_n)}\right)|m\alpha\rangle\langle p\mu| + \\
&- i\sum_{\substack{m\neq n\\ \alpha\beta}}\sum_{\substack{p\neq n\\ p\neq m\\ \mu}} B_{p\mu,n\beta}B_{n\beta,m\alpha}\left(\frac{e^{i(a_m-a_p)\tau_0}-1}{(a_m-a_p)(a_p-a_n)}\right)|p\mu\rangle\langle m\alpha| \\
&+ \tau_0\sum_{\substack{m\neq n\\ \alpha\beta\mu}} B_{m\alpha,n\beta}B_{n\beta,m\mu}\,\frac{|m\alpha\rangle\langle m\mu|}{a_m-a_n} + \\
&+ \tau_0\sum_{\substack{m\neq n\\ \alpha\beta\mu}} B_{m\mu,n\beta}B_{n\beta,m\alpha}\,\frac{|m\mu\rangle\langle m\alpha|}{a_m-a_n}
\end{aligned}
\tag{3.40}
$$

We note that we can write

$$
\frac{\partial U_0(\vec{\tau}_1)}{\partial\tau_2}\,U_0^{-1}(\vec{\tau}_1) = e^{i\Delta_1\tau_1}\,\frac{\partial U_0(\vec{\tau}_2)}{\partial\tau_2}\,U_0^{-1}(\vec{\tau}_2)e^{-i\Delta\tau_1} \tag{3.41}
$$

We now define Δ_2 through the equation

$$i \frac{\partial U_0(\vec{\tau}_2)}{\partial \tau_2} U_0^{-1}(\vec{\tau}_2) = -\Delta_2 \tag{3.42}$$

This can also be written as

$$\frac{\partial U_0(\vec{\tau}_1)}{\partial \tau_2} U_0^{-1}(\vec{\tau}_1) = e^{i\Delta_1\tau_1}(i\Delta_2)e^{-i\Delta_1\tau_1} \tag{3.43}$$

We thus obtain

$$\begin{aligned} U_2(\vec{\tau}_0)U_0^{-1}(\vec{\tau}_0) = {} & U_2(\vec{\tau}_1)U_0^{-1}(\vec{\tau}_1) - i\tau_0 e^{i\Delta_1\tau_1}\Delta_2 e^{-i\Delta_1\tau_1} + i\tau_0\left(i\frac{\partial U_1(\vec{\tau}_1)}{\partial \tau_1} + \Delta_1 U_1(\vec{\tau}_1)\right)\cdot \\ & \cdot U_0^{-1}(\vec{\tau}_1) + \int_0^{\tau_0} d\lambda[\mathcal{S}(\lambda),\Delta_1] + i\mathcal{S}(\tau_0) - \frac{1}{2}\mathcal{S}^2(\tau_0) + \\ & - \frac{1}{2}\int_0^{\tau_0} d\lambda[\dot{\mathcal{S}}(\lambda),\mathcal{S}(\lambda)] \end{aligned} \tag{3.44}$$

The secular part of this expression can be extracted explicitly. We have

$$\begin{aligned} \text{Sec } U_2(\vec{\tau}_0)U_0^{-1}(\vec{\tau}_0) = {} & -i\tau_0 e^{i\Delta_1\tau_1}\Delta_2 e^{-i\Delta_1\tau_1} + i\tau_0\left(i\frac{\partial U_1(\vec{\tau}_1)}{\partial \tau_1} + \right. \\ & \left. + \Delta_1 U_1(\vec{\tau}_1)\right)U_0^{-1}(\vec{\tau}_1) - i\tau_0 \sum_{\substack{m\neq n \\ \alpha\beta \\ \nu}} B_{m\alpha,n\beta}B_{n\beta,n\nu} \\ & \cdot \frac{|m\alpha\rangle\langle n\nu|}{(a_m-a_n)} - i\tau_0 \sum_{\substack{m\neq n \\ \alpha\beta \\ \nu}} B_{n\nu,n\beta}B_{n\beta,m\alpha} \frac{|n\nu\rangle\langle m\alpha|}{(a_m-a_n)} + \\ & - \frac{\tau_0}{2} \sum_{\substack{m\neq n \\ \alpha\beta\mu}} B_{m\alpha,n\beta}B_{n\beta,m\mu} \frac{|m\alpha\rangle\langle m\mu|}{a_m-a_n} + \\ & - \frac{\tau_0}{2} \sum_{\substack{m\neq n \\ \alpha\beta\mu}} B_{m\mu,n\beta}B_{n\beta,m\alpha} \frac{|m\mu\rangle\langle m\alpha|}{a_m-a_n} \end{aligned} \tag{3.45}$$

The time dependence of Δ_2 that is obtained in (3.43) is then obtained as

$$e^{i\Delta_1\tau_1}\Delta_2 e^{-i\Delta_1\tau_1} = \left(i\,\frac{\partial U_1(\vec{\tau}_1)}{\partial\tau_1} + \Delta_1 U_1(\vec{\tau}_1)\right)U_0^{-1}(\vec{\tau}_1) +$$

$$- \sum_{\substack{m\neq n\\ \alpha\beta\\ \nu}} B_{m\alpha,n\beta}B_{n\beta,n\nu}\,\frac{|m\alpha\rangle\ \langle n\nu|}{(a_m-a_n)} - \sum_{\substack{m\neq n\\ \alpha\beta\\ \nu}} B_{n\nu,n\beta}B_{n\beta,m\alpha}\ \cdot$$

$$\cdot\ \frac{|n\nu\rangle\ \langle m\alpha|}{(a_m-a_n)} + \frac{i}{2}\sum_{\substack{m\neq n\\ \alpha\beta\mu}} B_{m\alpha,n\beta}B_{n\beta,m\mu}\,\frac{|m\alpha\rangle\ \langle m\mu|}{(a_m-a_n)} +$$

$$+ \frac{i}{2}\sum_{\substack{m\neq n\\ \alpha\beta\mu}} B_{m\mu,n\beta}B_{n\beta,m\alpha}\,\frac{|m\mu\rangle\ \langle m\alpha|}{(a_m-a_n)} \tag{3.46}$$

We now introduce a new basis as

$$|m\alpha\rangle^* = \sum_{\gamma} C^m_{\alpha\gamma}|m\gamma\rangle \tag{3.47}$$

where

$$\Delta_1|m\alpha\rangle^* = \delta^m_\alpha|m\alpha\rangle^* \tag{3.48}$$

The degeneracy is removed in the new basis by the first-order perturbation. We note the following relation

$$A|m\alpha\rangle^* = a_m\sum_{\gamma} C^m_{\alpha\gamma}\ |m\gamma\rangle = a_m|m\alpha\rangle^* \tag{3.49}$$

The orthogonality of the new basis is expressed by

$$ {}^{*}\langle n\beta | m\alpha^{*}\rangle = \sum_{\gamma\nu} \bar{c}^{m}_{\beta}\; c^{m}_{\alpha\gamma}\langle m\mu | m\gamma\rangle $$

$$ = \sum_{\gamma} \bar{c}^{m}_{\beta\gamma} c^{m}_{\alpha\gamma} \tag{3.50} $$

In view of completeness, we can write

$$ \delta_{\alpha\beta} = \sum_{\gamma} c^{m}_{\alpha\gamma}\; c^{m+}_{\gamma\beta} \tag{3.51} $$

whence

$$ \sum_{\alpha\gamma} c^{m+}_{\mu\alpha}\; c^{m}_{\alpha\gamma}\; c^{m+}_{\gamma\beta} = c^{m+}_{\mu\beta} \tag{3.52} $$

Therefore, we also have

$$ \delta_{\mu\gamma} = \sum_{\alpha} c^{m+}_{\mu\alpha}\; c^{m}_{\alpha\gamma} \tag{3.53} $$

We can then write

$$ \sum_{\alpha} c^{m+}_{\mu\alpha} | m\alpha\rangle^{*} = \sum_{\alpha}\sum_{\gamma} c^{m+}_{\mu\alpha}\; c^{m}_{\alpha\gamma} | m\gamma\rangle $$

$$ = \sum_{\gamma} \delta_{\mu\gamma}\; | m\gamma\rangle = | m\mu\rangle \tag{3.54} $$

The new basis can be written as

$$ | m\mu\rangle = \sum_{\alpha} c^{m+}_{\mu\alpha}\; | m\alpha\rangle^{*} \tag{3.55} $$

The adjunct vector is then computed by

$$ \langle m\mu | = \sum_{\alpha} \overline{c^{m+}_{\mu\alpha}}\;\; {}^{*}\langle m\alpha |^{*} $$

$$ = \sum_{\alpha} c^{m}_{\alpha\mu}\; {}^{*}\langle m\alpha | \tag{3.56} $$

so that we have

$$\langle m\mu| = \sum_{\alpha} C^{m}_{\alpha\mu}{}^{*}\langle m\alpha| \tag{3.57}$$

The time-dependent Δ_2 can then be written as

$$e^{i\Delta_1\tau_1}\Delta_2 e^{-i\Delta_1\tau_1} = \left(i\frac{\partial U_1(\vec{\tau}_1)}{\partial\tau_1} + \Delta_1 U_1(\vec{\tau}_1)\right)U_0^{-1}(\vec{\tau}_1) +$$

$$-\sum_{\substack{m\neq n\\ \alpha\beta\\ \nu}} B_{m\alpha,n\beta}B_{n\beta,n\nu}\frac{|m\alpha\rangle\langle n\nu|}{(a_m-a_n)} - \sum_{\substack{m\neq n\\ \alpha\beta\\ \nu}} B_{n\nu,n\beta}B_{n\beta,m\alpha}\frac{|m\nu\rangle\langle m\alpha|}{(a_m-a_n)} +$$

$$+\frac{i}{2}\sum_{\substack{m\neq n\\ \alpha\beta\mu}}\sum_{\nu\gamma} B_{m\alpha,n\beta}B_{n\beta,m\mu}\frac{C^{m\,+}_{\alpha\nu}C^{m}_{\gamma\mu}}{(a_m-a_n)}\,|m\nu\rangle^{*}\;{}^{*}\langle m\gamma| +$$

$$+\frac{i}{2}\sum_{\substack{m\neq n\\ \alpha\beta\mu}}\sum_{\nu\gamma} B_{m\mu,n\beta}B_{n\beta,m\alpha}\frac{C^{m\,+}_{\mu\nu}C^{m}_{\gamma\alpha}}{(a_m-a_n)}\,|m\nu\rangle^{*}\;{}^{*}\langle m\gamma| \tag{3.58}$$

We now assume that the starred basis is made by nondegenerate eigenfunctions of Δ_1 . We can then let

$$i\frac{\partial U_1}{\partial\tau_1} + \Delta_1 U_1 = \sum_{\substack{m\neq n\\ \alpha\beta\\ \nu}} B_{m\alpha,n\beta}B_{n\beta,n\nu}\frac{|m\alpha\rangle\;\langle n\nu|}{(a_m-a_n)}\,U_0(\vec{\tau}_1)$$

$$+\sum_{\substack{m\neq n\\ \alpha\beta\\ \nu}} B_{n\nu,n\beta}B_{n\beta,m\alpha}\frac{|n\nu\rangle\;\langle m\alpha|}{(a_m-a_n)}\,U_0(\vec{\tau}_1) +$$

$$- \frac{i}{2} \sum_{\substack{m \neq n \\ \alpha\beta\mu}} \sum_{\nu \neq \gamma} B_{m\alpha,n\beta} B_{n\beta,m\mu} \frac{C_{\alpha}^{m^{+}} C_{\gamma}^{m}}{(a_m - a_n)} \; |m\nu\rangle^{*} \; {}^{*}\langle m\gamma| U_0(\vec{\tau}_1)$$

$$- \frac{i}{2} \sum_{\substack{m \neq n \\ \alpha\beta\mu}} \sum_{\nu \neq \gamma} B_{m\mu,n\beta} B_{n\beta,m\alpha} \frac{C^{m^{+}} C_{\gamma\alpha}^{m}}{(a_m - a_n)} \; |m\nu\rangle^{*} \; {}^{*}\langle m\gamma| U_0(\vec{\tau}_1) \quad (3.59)$$

where the quantity Δ_2 is given by

$$\Delta_2 = \frac{i}{2} \sum_{\substack{m \neq n \\ \alpha\beta\mu}} \sum_{\nu} B_{m\alpha,n\beta} B_{n\beta,m\mu} \frac{C_{\alpha\nu}^{m^{+}} C_{\nu\mu}^{m}}{(a_m - a_n)} \; |m\nu\rangle^{*} \; {}^{*}\langle m\nu| \; +$$

$$+ \frac{i}{2} \sum_{\substack{m \neq n \\ \alpha\beta\mu}} \sum_{\nu} B_{m\mu,n\beta} B_{n\beta,m\alpha} \frac{C_{\mu\nu}^{m^{+}} C_{\nu\alpha}^{m}}{(a_m - a_n)} \; |m\nu\rangle^{*} \; {}^{*}\langle m\nu| \qquad (3.60)$$

The expression for Δ_2 that has been obtained will be studied in some detail. In particular, we have

$$[\Delta_2, \Delta_1] = [\Delta_2, A] = 0 \qquad (3.61)$$

Thus, the starred basis is constituted by eigenfunctions of Δ_2 so that in order to go to higher-order theory another basis is unnecessary. Since we can write

$$\Delta_1 |m\alpha\rangle^{*} = \delta_{\alpha}^{m} |m\alpha\rangle^{*} \qquad (3.62)$$

we have

$$\Delta_1 = \sum_{\substack{m \\ \alpha\beta}} B_{m\alpha,m\beta} |m\alpha\rangle \; \langle m\beta| \qquad (3.63)$$

$$\Delta_1^{m} = \sum_{\alpha\beta} B_{m\alpha,m\beta} |m\alpha\rangle \; \langle m\beta| \qquad (3.64)$$

The adjoint statement is

$$\langle m\nu| \Delta_1^m = \sum_{\alpha\beta} B_{m\alpha,m\beta} \langle m\nu|m\alpha\rangle \langle m\beta|$$

$$= \sum_{\beta} B_{m\nu,m\beta} \langle m\beta| \tag{3.65}$$

which can be written as

$$\sum_{\beta} B_{m\nu,m\beta} \langle m\beta| = \langle m\nu| \Delta_1^m \tag{3.66}$$

Similarly, the matrix representation of the direct statement is given by

$$\Delta_1^m |m\nu\rangle = \sum_{\alpha\beta} B_{m\alpha,m\beta} |m\alpha\rangle \langle m\beta|m\nu\rangle$$

$$= \sum_{\alpha} B_{m\alpha,m\nu} |m\alpha\rangle \tag{3.67}$$

so that

$$\Delta_1^m |m\nu\rangle = \sum_{\alpha} B_{m\alpha,m\nu} |m\alpha\rangle \tag{3.68}$$

We notice the following relation

$$\sum_{\substack{m\neq n\\ \alpha\beta\\ \nu}} B_{m\alpha,n\beta} B_{n\beta,n\nu} \frac{|m\alpha\rangle \langle n\nu|}{a_m-a_n} + \sum_{\substack{m\neq n\\ \alpha\beta\\ \nu}} B_{n\nu,n\beta} B_{n\beta,m\alpha} \frac{|n\nu\rangle \langle m\alpha|}{a_m-a_n} =$$

$$= \sum_{m\neq n} B_{m\alpha,n\beta} \frac{|m\alpha\rangle \langle n\beta| \Delta_1^n}{a_m-a_n} + \sum_{\substack{m\neq n\\ \alpha\beta}} B_{n\beta,m\alpha} \frac{\Delta_1^n |n\beta\rangle \langle m\alpha|}{a_m-a_n} =$$

$$= \sum_{\substack{m\neq n\\ \alpha\beta}} B_{m\alpha,n\beta} \frac{|m\alpha\rangle \langle n\beta|\Delta_1 - \Delta_1|n\alpha\rangle \langle n\beta|}{(a_m - a_n)}$$

$$= \left[\sum_{\substack{m\neq n\\ \alpha\beta}} B_{m\alpha,n\beta} \frac{|m\alpha\rangle \langle n\beta|}{(a_m - a_n)} , \Delta_1 \right] \qquad (3.69)$$

We can then write the first-order equation as

$$i \frac{\partial U_1}{\partial \tau_1} + \Delta_1 U_1 = \left[\sum_{\substack{m\neq n\\ \alpha\beta}} B_{m\alpha,n\beta} \frac{|m\alpha\rangle \langle n\beta|}{(a_m - a_n)} , \Delta_1 \right] U_0(\vec{\tau}_1) +$$

$$- \frac{i}{2} \sum_{\substack{m\neq n\\ \alpha\beta\mu}} \sum_{\nu\neq\gamma} B_{m\alpha,n\beta} B_{n\beta,m\mu} \frac{C^{m^+}_{\alpha\nu} C^m_{\gamma\mu}}{(a_m - a_n)} |m\nu\rangle^* \, {}^*\langle m\gamma| U_0(\vec{\tau}_1) +$$

$$- \frac{i}{2} \sum_{\substack{m\neq n\\ \alpha\beta\mu}} \sum_{\nu\neq\gamma} B_{m\mu,n\beta} B_{n\beta,m\alpha} \frac{C^{m^+}_{\mu\nu} C^m_{\gamma\alpha}}{(a_m - a_n)} |m\nu\rangle^* \, {}^*\langle m\gamma| U_0(\vec{\tau}_1) \qquad (3.70)$$

A slight rearrangement yields

$$i \frac{\partial U_1}{\partial \tau_1} + \Delta_1 U_1 = \left[\sum_{\substack{m\neq n\\ \alpha\beta}} B_{m\alpha,n\beta} \frac{|m\alpha\rangle \langle n\beta|}{(a_m - a_n)} , \Delta_1 \right] U_0(\vec{\tau}_1) +$$

$$-i \sum_{\substack{m\neq n\\ \alpha\beta\mu}} \sum_{\nu\neq\gamma} B_{m\alpha,n\beta} B_{n\beta,m\mu} \frac{C^{m^+}_{\alpha\nu} C^m_{\gamma\mu}}{(a_m - a_n)} |m\nu\rangle^* \, {}^*\langle m\gamma| U_0(\vec{\tau}_1) \qquad (3.71)$$

We can then obtain an explicit representation for the first-order operator.

$$iU_1(\vec{\tau}_1) = ie^{i\Delta_1\tau_1}U_1(\vec{\tau}_2)+\sum_{\substack{m\neq n\\ \alpha\beta}} \frac{B_{m\alpha,n\beta}}{(a_m-a_n)}\int_0^{\tau_1}d\lambda e^{i\Delta_1\lambda}\Big[|m\alpha\rangle\langle n\beta|,\Delta_1\Big]e^{-i\Delta_1\lambda}U_0(\vec{\tau}_1)+$$

$$-i\sum_{\substack{m\neq n\\ \alpha\beta\mu}}\sum_{\nu\neq\gamma} B_{m\alpha,n\beta}B_{n\beta,m\mu}\frac{C^{m+}_{\alpha\nu}C^{m}_{\gamma\mu}}{(a_m-a_n)}\int_0^{\tau_1} d\lambda e^{i\Delta_1\lambda}|m\nu\rangle^{**}\langle m\gamma|e^{-i\Delta_1\lambda}U_0(\vec{\tau}_1)$$

$$= ie^{i\Delta_1\tau_1}U_1(\vec{\tau}_2)+i\sum_{\substack{m\neq n\\ \alpha\beta}}\frac{B_{m\alpha,n\beta}}{(a_m-a_n)}\int_0^{\tau_1}d\lambda\frac{\partial}{\partial\lambda}\Big[e^{i\Delta_1\lambda}|m\alpha\rangle\langle n\beta|e^{-i\Delta_1\lambda}\Big]U_0(\vec{\tau}_1)+$$

$$-i\sum_{\substack{m\neq n\\ \alpha\beta\mu}}\sum_{\nu\neq\gamma} B_{m\alpha,n\beta}B_{n\beta,m\mu}\frac{C^{m+}_{\alpha\nu}C^{m}_{\gamma\mu}}{(a_m-a_n)}\int_0^{\tau_1}d\lambda e^{i(\delta^m_\nu-\delta^m_\gamma)\lambda}|m\nu\rangle^{**}\langle m\gamma|U_0(\vec{\tau}_1) \tag{3.72}$$

After some rearrangement, we have

$$U_1(\vec{\tau}_1) = e^{i\Delta_1\tau_1}U_1(\vec{\tau}_2) + \sum_{\substack{m\neq n\\ \alpha\beta}}\frac{B_{m\alpha,n\beta}}{(a_m-a_n)}\Big[e^{i\Delta_1\tau_1}|m\alpha\rangle\langle n\beta|e^{-i\Delta_1\tau_1}-|m\alpha\rangle\langle n\beta|\Big]U_0(\vec{\tau}_1)+$$

$$-\sum_{\substack{m\neq n\\ \alpha\beta\mu}}\sum_{\nu\neq\gamma} B_{m\alpha,n\beta}B_{n\beta,m\mu}\frac{C^{m+}_{\alpha\nu}C^{m}_{\gamma\mu}}{(a_m-a_n)}\left(\frac{e^{i(\delta^m_\nu-\delta^m_\gamma)\tau_1}-1}{i(\delta^m_\nu-\delta^m_\gamma)}\right)|m\nu\rangle^{**}\langle m\gamma|U_0(\vec{\tau}_1) \tag{3.73}$$

This expression can be manipulated into the form

$$U_1(\vec{\tau}_0) = e^{i\Delta_1\tau_1}U_1(\vec{\tau}_2) + \sum_{\substack{m\neq n\\ \alpha\beta}}\frac{B_{m\alpha,n\beta}}{(a_m-a_n)}e^{i\Delta_1\tau_1}|m\alpha\rangle\langle n\beta|e^{-i\Delta_1\tau_1}U_0(\vec{\tau}_1) +$$

$$-\sum_{\substack{m\neq n\\ \alpha\beta\mu}}\sum_{\nu\neq\gamma} B_{m\alpha,n\beta}B_{n\beta,m\mu}\frac{C^{m+}_{\alpha\nu}C^{m}_{\gamma\mu}}{(a_m-a_n)}\left(\frac{e^{i(\delta^m_\nu-\delta^m_\gamma)\tau_1}-1}{i(\delta^m_\nu-\delta^m_\gamma)}\right)|m\nu\rangle^{**}\langle m\gamma|U_0(\vec{\tau}_1)+$$

$$-\sum_{\substack{m\neq n\\ \alpha\beta}} B_{m\alpha,n\beta}\,\frac{e^{-i(a_m-a_n)\tau_0}}{(a_m-a_n)}\,|m\alpha\rangle\,\langle n\beta|U_0(\vec{\tau}_1) \tag{3.74}$$

We can therefore write.

$$U_1(\vec{\tau}_0) = e^{i\Delta_1\tau_1}U_1(\vec{\tau}_2)+\sum_{\substack{m\neq n\\ \alpha\beta}} \frac{B_{m\alpha,n\beta}}{(a_m-a_n)}\sum_{\nu\gamma} C^{m+}_{\alpha\nu}C^{n}_{\gamma\beta}e^{i(\delta^m_\nu-\delta^n_\gamma)\tau_1}|m\nu\rangle^{**}\langle n\gamma|U_0(\vec{\tau}_1) +$$

$$-\sum_{\substack{m\neq n\\ \alpha\beta}} B_{m\alpha,n\beta}\,\frac{e^{-i(a_m-a_n)\tau_0}}{(a_m-a_n)}\sum_{\nu\gamma} C^{m+}_{\alpha\nu}C^{n}_{\gamma\beta}\;|m\nu\rangle^{**}\langle n\gamma|U_0(\vec{\tau}_1)$$

$$-\sum_{\substack{m\neq n\\ \alpha\beta\mu}}\sum_{\nu\neq\gamma} B_{m\alpha,n\beta}B_{n\beta,m\mu}\,\frac{C^{m+}_{\alpha\nu}C^{m}_{\gamma\mu}}{(a_m-a_n)}\left(\frac{e^{i(\delta^m_\nu-\delta^m_\gamma)\tau_1}-1}{i(\delta^m_\nu-\delta^m_\gamma)}\right)|m\nu\rangle^{**}\langle m\gamma|U_0(\vec{\tau}_1) \tag{3.75}$$

We return to the original noninteraction representation operator and see

$$V_1(\vec{\tau}_0) = e^{iA\tau_0}e^{i\Delta_1\tau_1}U_1(\vec{\tau}_2)+\sum_{\substack{m\neq n\\ \alpha\beta\\ \nu\gamma}} \frac{B_{m\alpha,n\beta}}{(a_m-a_n)}\,C^{m+}_{\alpha\nu}C^{n}_{\gamma\beta}\left[e^{i(a_m-a_n)\tau_0}e^{i(\delta^m_\nu-\delta^n_\gamma)\tau_1}-1\right]\cdot$$

$$\cdot\;|m\nu\rangle^{**}\langle n\gamma|V_0(\vec{\tau}_0)\;+$$

$$+\,i\sum_{\substack{m\neq n\\ \alpha\beta\mu}}\sum_{\nu\neq\gamma} B_{m\alpha,n\beta}B_{n\beta,m\mu}\,\frac{C^{m+}_{\alpha\nu}C^{m}_{\gamma\mu}}{(a_m-a_n)}\left[\frac{e^{i(\delta^m_\nu-\delta^m_\gamma)\tau_1}-1}{(\delta^m_\nu-\delta^m_\gamma)}\right]|m\nu\rangle^{**}\langle m\gamma|V_0(\vec{\tau}_0) \tag{3.76}$$

We also have

$$V_0(\vec{\tau}_0) = e^{iA\tau_0} e^{i\Delta_1\tau_1} e^{i\Delta_2\tau_2} V_0(\vec{\tau}_3). \tag{3.77}$$

This gives a compact representation of the uniformized result.

It is of interest to note the expression for the secularity in second order. It is given by

$$\begin{aligned}
U_2(\vec{\tau}_0)U_0^{-1}(\vec{\tau}_0) &= U_2(\vec{\tau}_1)U_0^{-1}(\vec{\tau}_1) - \sum_{\substack{m\neq n\\ \alpha\beta\\ \nu}} B_{m\alpha,n\beta}B_{n\beta,n\nu}\left(\frac{e^{-i(a_m-a_n)\tau_0}-1}{(a_m-a_n)^2}\right)|m\alpha\rangle\langle n\nu| \; + \\
&+ \sum_{\substack{m\neq n\\ \alpha\beta\\ \nu}} B_{n\nu,n\beta}B_{n\beta,n\alpha}\left(\frac{e^{i(a_m-a_n)\tau_0}-1}{(a_m-a_n)^2}\right) |n\nu\rangle \langle m\alpha| \; + \\
&- \sum_{\substack{m\neq n\\ \alpha\beta}} B_{m\alpha,n\beta}\left(\frac{e^{-i(a_m-a_n)\tau_0}}{a_m-a_n}\right)|m\alpha\rangle \langle n\beta| + \\
&- \frac{1}{2}\sum_{\substack{m\neq n\\ \alpha\beta}} \sum_{\substack{m'\neq n'\\ \alpha'\beta'}} B_{m'\alpha,n\beta}B_{m'\alpha';n'\beta'}\left(\frac{e^{-i(a_m-a_n)\tau_0}-1}{a_m-a_n}\right)\left(\frac{e^{-i(a_{m'}-a_{n'})\tau_0}-1}{a_{m'}-a_{n'}}\right)\\
&\cdot \; |m\alpha\rangle \langle n\beta| \, |m'\alpha'\rangle \langle n'\beta'|
\end{aligned} \tag{3.78}$$

The main result (Eq. (3.77) shows a modified Schwinger-Dyson expansion is the propagation operator which is uniformly valid, providing double and triple summations that have been considered are really convergent. The main results are explicit formulae for the secular parts which can be extracted without ambiguity. The main tool employed has been the completeness of the eigenfunctions of the unperturbed operators. Note that if the starred basis still had contained degeneracy relative to Δ_1 , then the procedure outlined here to remove the degeneracy with Δ_1 could be repeated with Δ_2 or higher-order Δ's until the degeneracy is removed.

CHAPTER V

TECHNIQUE FOR TREATING THE LONG-RANGE KERNEL IN THE FLUX EQUATIONS

The equation for the cosmic ray flux in a random magnetic field can be modeled by

$$\frac{\partial f}{\partial t} = -\epsilon \int_0^t d\lambda \, \frac{f(t-\lambda)}{1+\lambda} \tag{1}$$

where $f(t)$ is the flux which is considered here to be only a function of time. The integral term on the right represents the effect of the random field on the flux; the parameter, ϵ, which measures the coupling is considered small. The problem is to determine the long-time (or near eqauilibrium) behavior of the flux, which then can be used to determine the transport properties of the cosmic ray density. Perturbation or iteration techniques produce expansions of $f(t)$ in ϵ which are nonuniform for large time, and the standard adiabatic approximation fails because of the long range of the kernel which decays only $\sim 1/\lambda$ for large λ.

The type of equation modeled by Eq. (1) appears often in studies of kinetic theory, but usually with a kernel, $K(\lambda)$, which is short-ranged due to the typically short-ranged interactions which are studied. The adiabatic approximation can then be made to determine the leading long-time behavior of $f(t)$ Since the coupling parameter is small, we assume that $f(t)$ changes slowly compared to the time scale over which the kernel, which represents the short-ranged interaction, decays. Then a good approximation to $f(t)$ is given by

$$\frac{\partial f}{\partial t} = -\epsilon \Delta f \tag{2}$$

where Δ is a constant given by

$$\Delta = \int_0^\infty d\lambda \; K(\lambda) \tag{3}$$

The flux decays exponentially to zero with a rate determined by Δ. Clearly, this approach fails for $K(\lambda) = 1/1+\lambda$, since Δ

becomes undefined. The problem with Δ is directly related to the decay of $K(\lambda)$ for large λ, which is too slow for $K(\lambda)$ to be integrable in the sense of Eq. (3).

The long tail in $K(\lambda)$ for the cosmic ray flux is due to those particles which have very small velocity, compared to a typical particle, parallel to the mean magnetic field. Thus, although their velocity may be high, their translation in space is quite slow and their interaction with the random magnetic field in a given region of space can last arbitrarily long. In previous attempts to construct kinetic theories for cosmic rays, the adiabatic approximation has been applied, in spite of its failings, with artificial cutoffs in the upper limit of Eq. (3) or by ignoring the particles in that region of phase space which have pitch-angle near 90° and which contribute the tail in $K(\lambda)$.

In this chapter, we present several approaches which give the long-time behavior of $f(t)$. One of these approaches, which involves the time scale extension technique, is general enough to be applied to other kernels which may be more complicated but which have the feature that for large λ, $K(\lambda) \sim 1/\lambda$. This approach is ideally suited to the true cosmic ray flux equations.

1. THE LAPLACE TRANSFORM

We can obtain an indication of the likely large-time behavior of $f(t)$ by studying its Laplace transform near the origin in the transform variable, ω. From Eq. (1), we find

$$\tilde{f}(\omega) = \frac{f(0)}{\omega + \epsilon e^{\omega} E_1(\omega)} \tag{1.1}$$

where $E_1(\omega)$ is the exponential integral

$$E_1(\omega) = \int_{\omega}^{\infty} ds\, \frac{e^{-s}}{s} \tag{1.2}$$

$E_1(\omega)$ has a branch line extending from the origin along the negative real axis to infinity. Near the origin, it is given by

$$E_1(\omega) = -\gamma - \ln \omega - \sum_{n=1}^{\infty} \frac{(-1)^n \omega^n}{n n!}, \qquad (|\arg \omega| < \pi) \tag{1.3}$$

and for large ω it behaves asymptotically as

$$E_1(\omega) \sim \frac{e^{-\omega}}{\omega}\left[1 - \frac{1}{\omega} + \frac{2}{\omega^2} \cdots\right] \qquad \left(|\arg \omega| < \frac{3\pi}{2}\right) \tag{1.4}$$

γ is Euler's constant; i.e., $\gamma \simeq .577\ldots$. The Laplace transform, $\tilde{f}(\omega)$, has a pair of poles very near the origin in the negative real half of the ω-plane. We ignore all other poles to the left of these, and also the contribution of the branch line, to determine the large-time behavior of $f(t)$. This procedure cannot give a proof of the behavior of $f(t)$ unless carried to much more detail than we have managed; however, it can serve as a guide to the kind of solution we should look for with other techniques.

We look for solutions to

$$\omega + \epsilon e^{\omega} E_1(\omega) = 0 \tag{1.5}$$

with

$$\omega = R e^{i\theta} \tag{1.6}$$

and with $R \ll 1$. From Eq. (1.3), we let $E_1(\omega) = -\ell n\, \Gamma\omega$ where $\Gamma = \ell n\, \gamma$. The real and imaginary parts of Eq. (1.5) can be manipulated to obtain

$$R^2 = \epsilon^2 e^{2R\cos\theta}\left[(\ell n\, \Gamma R)^2 + \theta^2\right] \tag{1.7}$$

and

$$\theta = \alpha + R\sin\theta \tag{1.8}$$

where

$$\cos\alpha = \frac{\ell n\, \Gamma R}{\sqrt{(\ell n\, \Gamma R)^2 + \theta^2}} \tag{1.9}$$

For small R and $|\theta| < \pi$, Eqs. (1.7) and (1.8) have a pair of solutions which can be written approximately as

$$\omega_{\pm} = \epsilon|\ell n\, \epsilon| e^{\pm i\pi\left(1 - \frac{1}{|\ell n\, \epsilon|}\right)} \tag{1.10}$$

Thus, for large t , we expect

$$f(t) \sim e^{-\epsilon|\ell n\, \epsilon| t}[A\cos\epsilon\pi t + B\sin\epsilon\pi t] \tag{1.11}$$

where A and B are constants.

Equation (1.11) has two surprising features. First, there are oscillations on an (ϵt)-time scale and, second, the solution is exponentially damped on a $\epsilon|\ell n\, \epsilon|t$-time scale. Since the damping is fast compared to the oscillations, the oscillations can only make a minor modification to the basic exponential solution. We can begin to see why the adiabatic solution can never give a good approximation to the long-time behavior. In that approximation, we effectively impose exponential damping on the slow (ϵt)-time scale but the damping coefficient becomes infinite. What that solution is trying to tell us is that the damping is

finite but on a faster time scale than (ϵt) . All hand-waving arguments which attempt to produce a finite damping coefficient on the (ϵt) scale actually carry us further from the correct long-time behavior since they actually slow down the exponential damping even further. Relative to the cosmic ray problem, these attempts make the effects of the random field on the particles appear much weaker than they actually are.

2. THE OUTER EXPANSION

In this technique, we introduce two tricks which make it possible to find the long-time behavior of $f(t)$. In the first, we introduce a new unknown function, $g(t)$, through

$$f(t) = e^{\epsilon t} g(t) \tag{2.1}$$

Then

$$\frac{\partial g}{\partial t} + \epsilon g = -\epsilon \int_0^t d\lambda \, \frac{e^{-\epsilon\lambda}}{1+\lambda} \, g(t-\lambda) \tag{2.2}$$

In addition, we introduce

$$\tau = \epsilon t \quad , \qquad g\left(\frac{\tau}{\epsilon}\right) \equiv \chi(\tau) \tag{2.3}$$

to obtain

$$\frac{\partial \chi}{\partial \tau} + \chi = -e^{\epsilon} \int_{\epsilon}^{\epsilon+\tau} d\lambda \, \frac{e^{-\lambda}}{\lambda} \, \chi(\epsilon+\tau-\lambda) \tag{2.4}$$

Thus, Eq. (2.4) for χ is more amenable to the adiabatic approximation due to the exponential damping which we have forced into the kernel.

For large τ , in order to obtain the leading contribution to χ in an asymptotic expansion in ϵ , we write

$$\frac{\partial \chi}{\partial \tau} + \chi = -\int_{\epsilon}^{\infty} d\lambda \, \frac{e^{-\lambda}}{\lambda} \, \chi(\tau-\lambda) \tag{2.5}$$

The adiabatic approximation now gives

$$\frac{\partial \chi}{\partial \tau} + \left[1+E_1(\epsilon)\right]\chi = 0 \tag{2.6}$$

or

$$\chi(\tau) = e^{-[1+E_1(\epsilon)]\tau}\,\chi(0) \tag{2.7}$$

Then

$$f(t) = e^{-\epsilon|\ell n\ \epsilon|t}\, f(0) \tag{2.8}$$

In this case, the oscillations are missed, but since they are very slow in any case, this is not a serious shortcoming of this technique. In fact, we will now proceed to refine this technique to regain the oscillations; however, it is doubtful whether this refinement could be applied in the case of a more complicated kernel. In the next section, we will present a technique which gives both the proper damping and the oscillations and which seems more generally applicable.

We go back to Eq. (2.5) and assume the analyticity of $\chi(t)$ Thus, we can Taylor expand $\chi(\tau-\lambda)$ about $\chi(\tau)$. In this case,

$$\frac{\partial\chi}{\partial\tau} + \left[1+E_1(\epsilon)\right]\chi = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}\,\epsilon^n\alpha^{n-1}(\epsilon)\,\frac{\partial^n\chi}{\partial\tau^n} \tag{2.9}$$

where

$$\alpha^{n-1}(\epsilon) = (n-1)!\,\epsilon^{-n}\left[1+\epsilon+\frac{\epsilon^\tau}{2!}\ \cdots\ + \frac{\epsilon^{n-1}}{(n-1)!}\right. \tag{2.10}$$

We once again consider only the leading terms in ϵ to find

$$\frac{\partial\chi}{\partial\tau} + \left[1+E_1(\epsilon)\right]\chi = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\,\frac{\partial^n\chi}{\partial\tau^n} \tag{2.11}$$

We substitute

$$\chi = e^{-(1+E_1(\epsilon)-\nu)\tau}\chi(0) \tag{2.12}$$

to determine the correction, ν , to the basic $\epsilon|\ell n\ \epsilon|$ decay we

found from the adiabatic approximation. We find

$$\nu = \ln[\nu - E_1(\epsilon)] \tag{2.13}$$

Thus,

$$\nu = \ln |\ln \epsilon| \pm i\pi + O \frac{\nu}{|\ln \epsilon|} \tag{2.14}$$

Neglecting $\ln|\ln \epsilon|$ compared to $E_1(\epsilon)$, we find

$$\chi = e^{-(1+|\ln \epsilon| \pm i\pi)\tau} \chi(0) \tag{2.15}$$

and

$$f(t) = e^{-\epsilon|\ln \epsilon| t}\left[A \cos \epsilon\pi t + B \sin \epsilon\pi t\right] \tag{2.16}$$

in agreement with the Laplace transform evaluation.

3. LINEAR TIME-SCALE EXTENSION

In this section, we present another technique for obtaining an asymptotic expansion of $f(t)$ valid for large t. This technique has the advantage of being somewhat more rigorous than the approach in Section 2 and also considerably more adaptable to cases where the kernel may be more complicated.

We introduce an extension of Eq. (1) as follows

$$\frac{D\bar{f}(\tau_0,\tau_1)}{Dt} = -\epsilon \int_0^{\tau_0} d\lambda \, \frac{\bar{f}(\tau_0-\lambda,\tau_1-\alpha\lambda)}{1+\lambda} \tag{3.1}$$

where $\bar{f}(\tau_0,\tau_1)$ is a function of two independent variables, τ_0 and τ_1, with the additional condition that $\bar{f}(t,\alpha t)=f(t)$. The total derivative, D/Dt, is along a trajectory in the two-dimensional (τ_0,τ_1) space specified by

$$\frac{D}{Dt} = \frac{\partial}{\partial\tau_0} + \alpha\frac{\partial}{\partial\tau_1} \tag{3.2}$$

Notice that if we set $\tau_0 = t$ and $\tau_1 = \alpha t$ in Eq. (3.1), we regain Eq. (1). This is the sense in which we call Eq. (3.1) an extension of Eq. (1).

The parameter α is meant to be a function of ϵ which is small and which is to be chosen to remove the nonuniform behavior in time from the perturbation expansion of $\bar{f}$. If the kernel were short-ranged, we would choose $\alpha = \epsilon$ and then the slow time scale would be $\tau_1 = \epsilon t$ along the "restricted" trajectory. The results of uniformizing the perturbation expansion would be equivalent to the adiabatic treatment of the original equation for f discussed in Section 2. In the case of this kernel, we find $\Delta = \infty$ and therefore conclude that we must search for a different slow time scale which is, nevertheless, faster than ϵt. Thus, we leave α an unspecified function of ϵ which must, however, satisfy the following relationships

$$\alpha \underset{\epsilon \downarrow}{\to} 0 \quad ; \qquad \frac{\epsilon}{\alpha} \underset{\epsilon \downarrow}{\to} 0 \tag{3.3}$$

We insert an expansion of $\bar{f}$ into Eq. (3.1) which is given by

$$\bar{f} = \bar{f}_0 + \frac{\epsilon}{\alpha} \bar{f}_1 + \ldots \tag{3.4}$$

and equate the coefficients of equal powers of the relevant parameters. In doing this, we must be especially careful since we already know that the integral on the right side of Eq. (3.1) is not $O(1)$ but is some other formal order, α, which we have not yet determined. Thus, we have

$$\frac{\partial \bar{f}_0}{\partial \tau_0} = 0 \tag{3.5}$$

and

$$\alpha \frac{\partial \bar{f}_0}{\partial \tau_1} + \frac{\epsilon}{\alpha} \frac{\partial \bar{f}_1}{\partial \tau_0} + \epsilon \frac{\partial \bar{f}_1}{\partial \tau_1} =$$

$$= -\epsilon \int_0^{\tau_0} d\lambda \, \frac{\bar{f}_0(\tau_0 - \lambda, \tau_1 - \alpha\lambda)}{1 + \lambda} - \frac{\epsilon^2}{\alpha} \int_0^{\tau_0} d\lambda \, \frac{\bar{f}_1(\tau_0 - \lambda, \tau_1 - \alpha\lambda)}{1 + \lambda} \tag{3.6}$$

We wish to obtain an asymptotic expansion of $\bar{f}$ to $O(\epsilon/\alpha)$; therefore, we neglect the term $\epsilon(\partial f_1)/\partial \tau_1$ and also the second integral term. Then, from Eq. (3.5)

$$\alpha \frac{\partial \bar{f}_0}{\partial \tau_1} + \frac{\epsilon}{\alpha} \frac{\partial \bar{f}_1}{\partial \tau_0} = -\epsilon \int_0^{\infty} d\lambda \, \frac{\bar{f}_0(\tau_1 - \alpha\lambda)}{1 + \lambda} + \epsilon \int_{\tau_0}^{\infty} d\lambda \, \frac{\bar{f}_0(\tau_1 - \alpha\lambda)}{1 + \lambda} \tag{3.7}$$

Upon integrating over τ_0, we find

$$\frac{\epsilon}{\alpha}\,\bar{f}_1(\tau_0,\tau_1) = \frac{\epsilon}{\alpha}\,\bar{f}_1(0,\tau_1) + \epsilon\int_0^{\tau_0} ds \int_s^\infty d\lambda\, \frac{\bar{f}_0(\tau_1-\alpha\lambda)}{1+\lambda} +$$

$$- \tau_0\left[\alpha\,\frac{\partial\bar{f}_0}{\partial\tau_1} + \epsilon\int_0^\infty d\lambda\,\frac{\bar{f}_0(\tau_1-\alpha\lambda)}{1+\lambda}\right] \tag{3.8}$$

To remove the nonuniform secular behavior of $\bar{f}_1$, we set

$$\frac{\partial\bar{f}_0}{\partial\tau_1} = -\frac{\epsilon}{\alpha}\int_0^\infty d\lambda\,\frac{\bar{f}_0(\tau_1-\alpha\lambda)}{1+\lambda} \tag{3.9}$$

and, therefore,

$$\bar{f}_1(\tau_0,\tau_1) = \bar{f}_1(0,\tau_1) + \alpha\int_0^{\tau_0} ds\int_s^\infty d\lambda\,\frac{\bar{f}_0(\tau_1-\alpha\lambda)}{1+\lambda} \tag{3.10}$$

We determine α from Eq. (3.9) by setting

$$\bar{f}_0(0,\tau_1) = e^{\tau_1}\bar{f}_0(0,0) \tag{3.11}$$

$$\alpha = -\epsilon e^{\alpha}E_1(\alpha) \tag{3.12}$$

The roots of this equation which satisfy the conditions of Eq. (3.13) are exactly those which we found in Section 2 in our discussion of the Laplace transform of f ; they are given by Eq. (1.10). Notice that these roots have negative real parts; therefore, even though $\bar{f}_0$ grows exponentially with τ_1 , along the restricted trajectory where $\tau_1 = \alpha t$, $f_0(t)$ will decay exponentially as $\epsilon(\ell n\ \epsilon)t$. In addition, we find from Eq. (3.10)

$$\bar{f}_1(\tau_0,\tau_1) = \bar{f}_1(0,\tau_1) - e^{\tau_1+\alpha}\Big[E_2(\alpha(1+\tau_0))-E_2(\alpha)\Big] \tag{3.13}$$

where

$$E_2(z) = \int_1^\infty \frac{e^{-zt}}{t^2}\,dt \tag{3.14}$$

For small α, but large $\alpha\tau_0$, Eq. (3.13) is given approximately by

$$\bar{f}_1(\tau_0,\tau_1) = \bar{f}_1(0,\tau_1) - e^{\tau_1+\alpha}\left[\frac{e^{-\alpha(1+\tau_0)}}{\alpha(1+\tau_0)} - 1\right] \tag{3.15}$$

Thus, we have indeed uniformized the expansion of $\bar{f}$ to order $1/|\ell n\ \epsilon|$.

Along the restricted trajectory $\bar{f}(\tau_0,\tau_1) = f(t)$. Thus, we have

$$\frac{\partial f_0}{\partial t} - \alpha f_0 = 0 \tag{3.16}$$

and

$$f_0(t) = e^{\alpha t} f_0(0) \tag{3.17}$$

where α is given by Eq. (1.10). By insisting on the reality of $f_0(t)$, we find

$$f(t) = e^{-\epsilon(\ell n\ \epsilon)t}[A \cos \epsilon\pi t + B \sin \epsilon\pi t] \tag{3.18}$$

for

$$t \sim \frac{1}{\epsilon|\ell n\ \epsilon|} \tag{3.19}$$

4. CONCLUSION

Of the three methods developed here to obtain the leading large time behavior of $f(t)$, the time-scale extension seems the most easily adapted to other more complicated kernels. On the other hand, if the slow oscillations were, in some situation, considered unimportant, then the outer expansion technique could be equally well adapted.

CHAPTER VI

GENERALIZED LARMOR THEOREM

1. INTRODUCTION

a. Alfvén's Guiding Center

We split the world velocity of a charged particle in an electromagnetic field into a space-like and a time-like orthogonal 4-vectors (with two degrees of freedom each) using field strength tensor and its dual. For uniform $\vec{\mathcal{E}}$ and $\vec{B}$ fields, the (Lorentz invariant) lengths of each of these two velocities are constants of the motion and generalize the "parallel" and "perpendicular" energies in a constant magnetic field. The space-like velocity always contains the periodic (gyratory) part of the motion of the particle, while the time-like component is the world velocity of the particle's generalized guiding center.

The equations of motion for the two projected velocities are solved in a "standard" configuration and combined to give the general solution for the trajectory of a particle in an arbitrary uniform field configuration, with the help of a simply constructed Lorentz transformation that represents a general $\vec{\mathcal{E}} \times \vec{B}$ drift.

Alfvén's definition of a guiding center description of the behavior of a charged particle (relativistic, or not) in a uniform magnetic field follows naturally from basic properties of the particle's trajectory. Specifically, the magnitude of the velocity is conserved in time and the motion is easily visualized in terms of gyrations about a point which moves with constant velocity along the magnetic field. The cycle average position of the particle is well-defined and called the guiding center. The distance from the guiding center to the particle position is called the gyration radius. In situations where, in addition to the magnetic field, there are weak electric fields and/or small gradients and/or time variations in the magnetic field, the Alfvén[1] guiding center plus gyration vector approximation to the particle motion is valid. In this approximation, the

guiding center retains its usefulness; the first-order corrections to its motion are computed using standard perturbation theory in terms of the small field variations. Even to first order, this technique has been very useful in many applications (see, for example, Northrop and Teller[2] for a study of the trapped particles in the Earth's geomagnetic field). More recently, higher-order terms than the first have been calculated by Kruskal[3] for a general class of dynamical systems which include the charged particle in a field.

b. New Results

In this chapter, we re-examine the study of the guiding center in uniform and stationary electric and magnetic fields of arbitrary magnitudes and directions. We obtain a relativistic generalization of the guiding center in space-time which is identical, in the nonrelativistic limit in 3-space, to the Alfvén guiding center.

We employ an operator formalism using projection operators which are constructed out of the field strength tensor and its dual. The pair of projection operators which we use form a complete orthogonal set which allows us to project the world velocity of the particle into two parts which lie in orthogonal, 2-dimensional subspaces. One of these parts is the world velocity of the guiding center and is time-like. Thus, in any nonsingular field configuration, we avoid the problem of averaging over the particle trajectory to obtain the guiding center position by merely projecting away the oscillatory part of the particle's velocity; the part that is left represents the (time-like) 4-velocity of the guiding center.

The operator formalism constructed in this report is applicable to any uniform field configuration except the singular case, $\vec{\mathcal{E}}\cdot\vec{B} = 0$ and $B^2 - \mathcal{E}^2 = 0$, where the operators become undefined.

c. Notation

Gaussian units are used throughout this paper. The transposed matrix is denoted by a tilde and a prime is used for a Lorentz-transformed quantity. The determinant of the matrix A is written det A . The dual of a tensor, defined by Eq. (1.3) is denoted by a star. The world velocity, u , is written as

$$u = (\gamma\vec{v}, ic\gamma) \quad , \qquad \gamma = [1 - v^2/c^2]^{-1/2} \tag{1.1}$$

and the proper time τ is related to laboratory time by

$$\gamma dt = d\tau \tag{1.2}$$

We define the dual f^* of a skew tensor f as

$$f^*_{\mu\nu} \equiv \frac{1}{2}\,\epsilon_{\mu\nu\sigma\eta} f_{\sigma\eta} \quad , \qquad f^{**} = f \tag{1.3}$$

The dual tensor is sometimes defined as

$$F^{*\prime}_{\mu\nu} = (1/2i)\epsilon_{\mu\nu\lambda\rho} f_{\lambda\rho} \tag{1.4}$$

Then

$$(f^{*\prime})^{*\prime} = -f \tag{1.5}$$

A 3-vector is denoted by $\vec{A}$ and its magnitude by

$$A = (\vec{A}\cdot\vec{A})^{1/2} \tag{1.6}$$

The signum function, sgn (x) , is defined by

$$\mathrm{sgn}\,(x) = \begin{cases} +1 & x > 0 \\ -1 & x < 0 \end{cases} \tag{1.7}$$

2. ELEMENTARY PROJECTION OPERATOR FORMALISM

We use the motion of a nonrelativistic charged particle in a uniform magnetic field to motivate the general projection operator formalism presented in the next section. Standard analysis is reproduced with (a special case of) our technique which projects the behavior of the guiding center and of the gyration vector from the equations of motion and thus facilitates carrying out their solution. To this end, we write the equation of motion in terms of (a 3×3 submatrix of) the electromagnetic field tensor (see Eq. (3.2)). The properties of the particle motion are then characterized through the algebraic properties of the frequency matrix. The velocity vector, $\vec{v}$, satisfies

$$\frac{d}{dt}\vec{v} = \underline{\Omega}\cdot\vec{v} \quad , \qquad \underline{\Omega} = \frac{e}{m_0 c}\begin{pmatrix} 0 & B_3 & -B_2 \\ -B_3 & 0 & B_1 \\ B_2 & -B_1 & 0 \end{pmatrix} \tag{2.1}$$

The useful properties of the field strength tensor are:

(A) The matrix $\underline{\Omega}$ is antisymmetric. Thus,

$$\underline{\Omega} = -\underline{\widetilde{\Omega}} \tag{2.2}$$

and therefore $\mathrm{Tr}\,\underline{\Omega} = \det\underline{\Omega} = 0$. We also have the representation

$$\Omega_{ij} = \epsilon_{ijk}\omega_B\beta_k \quad , \qquad \beta_k = B_k/B \quad , \qquad \vec{\omega}_B = \frac{e\vec{B}}{m_0 c} \tag{2.3}$$

Furthermore, $\exp(\underline{\Omega}t)$ is an orthogonal matrix which represents a rotation about the direction $\vec{\beta}$ with rotation angle (Bt)

$$\exp(\underline{\Omega}t) = R(\vec{\beta},Bt) = \widetilde{R}(\vec{\beta},-Bt) \tag{2.4}$$

(B) The matrix $\underline{\Omega}\cdot\underline{\widetilde{\Omega}}/\omega_B^2$ is a projection. Thus,

$$N \equiv -\frac{1}{\omega_B^2}\,\underline{\Omega}\cdot\underline{\Omega} = \underline{I} - \vec{\beta}\vec{\beta} \ , \qquad \omega_B^2 = \frac{1}{2}\,\mathrm{Tr}\,\underline{\Omega}\cdot\underline{\widetilde{\Omega}} \tag{2.5}$$

is a projection operator (normal to the magnetic field) since it is idempotent and symmetric

$$\underset{\sim}{N} \cdot \underset{\sim}{N} = \underset{\sim}{N} \quad , \qquad \tilde{\underset{\sim}{N}} = \underset{\sim}{N} \tag{2.6}$$

It follows that N is positive definite

$$N \geq 0 \tag{2.7}$$

and that

$$\underset{\sim}{P} \equiv \underset{\sim}{I} - \underset{\sim}{N} = \vec{\beta}\vec{\beta} \tag{2.8}$$

is a projection that satisfies the commutativity and orthogonality rules

$$\underset{\sim}{N} \cdot \underset{\sim}{P} = \underset{\sim}{P} \cdot \underset{\sim}{N} = 0 \tag{2.9}$$

(C) <u>The matrix</u> $\underset{\sim}{\Omega}$ <u>is "parallel" to</u> $\underset{\sim}{N}$ <u>and "orthogonal" to</u> $\underset{\sim}{P}$. Thus,

$$\underset{\sim}{N} \cdot \underset{\sim}{\Omega} = \underset{\sim}{\Omega} \cdot \underset{\sim}{N} = \Omega \quad , \qquad \underset{\sim}{P} \cdot \underset{\sim}{\Omega} = \underset{\sim}{\Omega} \cdot \underset{\sim}{P} = 0 \tag{2.10}$$

We can now show that the basic properties of the particle motion reflect our algebraic rules. For this purpose, we use the two projection operators introduced, $\underset{\sim}{N}$ and $\underset{\sim}{P}$, to decompose the velocity vector into two mutually orthogonal parts

$$\vec{v}_{\perp} \equiv \underset{\sim}{N} \cdot \vec{v} \quad , \qquad \vec{v}_{\parallel} \equiv \vec{v} - \vec{v}_{\perp} = \underset{\sim}{P} \cdot \vec{v} \tag{2.11}$$

each satisfying Eq. (2.1) because of (2.10).

We obtain the standard particle properties in parallel with the matrix properties just discussed.

(A) <u>Conservation of kinetic energy</u>. The quantities v^2 , $v_{\perp}^2$, $v_{\parallel}^2$ are conserved because of (2.2), e.g.,

$$\frac{1}{2} \frac{d}{dt} (v^2) = \vec{v} \cdot \frac{d\vec{v}}{dt} = \vec{v} \cdot \underset{\sim}{\Omega} \cdot \vec{v} = 0 \tag{2.12}$$

Furthermore, <u>the noninertial effect of the constant magnetic field is eliminated by transforming to a reference frame rotating with angular frequency</u> ω_B (Larmor theorem). This immediate consequence of (2.4) is generalized in the next section to include

relativistic motion as well as electric fields.

(B) <u>Periodicity of $\vec{v}_\perp$ with angular frequency ω_B</u>. Since $\vec{v}_\perp$ satisfies (2.1), we have, using the projection property (2.5),

$$\frac{d^2}{dt^2}\,\vec{v}_\perp = \underline{\Omega}\cdot\frac{d}{dt}\,\vec{v}_\perp = \underline{\Omega}\cdot\underline{\Omega}\cdot\vec{v}_\perp = -\omega_B^2\,\underline{N}\cdot\vec{v}_\perp = -\omega_B^2\vec{v}_\perp \qquad (2.13)$$

The vector $\vec{v}_\perp$ is harmonic because of (2.7).

(C) <u>Inertial motion of the guiding center</u>. Using (2.10)

$$\frac{d}{dt}\,\vec{v}_\parallel = \underline{P}\cdot\underline{\Omega}\cdot\vec{v} = 0 \qquad (2.14)$$

Thus, the integral of $\vec{v}_\parallel, \Delta\vec{x}_\parallel$ represents inertial motion. Integration of

$$\vec{v}_\parallel = \vec{v} + \frac{1}{\omega_B^2}\,\underline{\Omega}\cdot\underline{\Omega}\cdot\vec{v} = \vec{v} + \frac{1}{\omega_B^2}\,\underline{\Omega}\cdot\frac{d}{dt}\,\vec{v} \qquad (2.15)$$

motivates the definition of the guiding center by

$$\Delta\vec{x}_\parallel = \vec{x} + \frac{1}{\omega_B^2}\,\underline{\Omega}\cdot\vec{v} = \vec{x} - \frac{\vec{\omega}_B\times\vec{v}}{\omega_B^2} \equiv \Delta\vec{x}_{GC} \qquad (2.16)$$

The magnitude of the gyration vector, $\vec{x}_g \equiv \Delta\vec{x}_\perp$, is the usual radius of gyration r_g and is given by

$$r_g = |\vec{x}_\perp| = v_\perp/\omega_B \qquad (2.17)$$

Combining the integration of (2.13) and (2.14), we obtain useful representations for the particle trajectory

$$\vec{v}(t) = (\underline{P} + \underline{N}\cos\omega_B t + \frac{1}{\omega_B}\,\underline{\Omega}\sin\omega_B t)\cdot\vec{v}_0 \qquad (2.18)$$

$$\Delta\vec{x}(t) = \vec{v}_{\parallel 0}t + \left(\frac{\sin\omega_B t}{\omega_B}\,\underline{I} - \frac{\cos\omega_B t - 1}{\omega_B^2}\,\underline{\Omega}\right)\cdot\vec{v}_{\perp 0} \qquad (2.19)$$

The projections $\underline{\underline{P}}$ and $\underline{\underline{N}}$ can be eliminated in favor of $\underline{\Omega}$ by

using (2.8) and (2.5). In the next section, we show that the inclusion of relativity and electric fields is obtained by a straightforward analogy.

3. GENERAL FORMALISM

The Lorentz equations of motion of a relativistic charged particle in an electromagnetic field are

$$\frac{d}{d\tau} u = \underset{\sim}{\omega} \cdot u \tag{3.1}$$

where ω is the antisymmetric field strength tensor

$$\underset{\sim}{\omega} = \frac{e}{m_0 c} \begin{pmatrix} 0 & B_3 & -B_2 & -i\mathcal{E}_1 \\ -B_3 & 0 & B_1 & -i\mathcal{E}_1 \\ B_2 & -B_1 & 0 & -i\mathcal{E}_3 \\ i\mathcal{E}_1 & i\mathcal{E}_2 & i\mathcal{E}_3 & 0 \end{pmatrix} = -\underset{\sim}{\tilde{\omega}} \tag{3.2}$$

From Eq. (1.3), ω^* is given by the substitutions $B_i \leftrightarrow -i\mathcal{E}_i$. From ω and ω^* we can construct two invariants and two useful tensor relations

$$\det \omega = \det \omega^* = -\left(\frac{e}{m_0 c}\right)^4 (\vec{\mathcal{E}}\cdot\vec{B})^2 \tag{3.3}$$

$$\omega^2 \equiv \frac{1}{2}\, \omega_{\mu\nu}\omega_{\mu\nu} = \frac{1}{2}\, \omega^*_{\mu\nu}\omega^*_{\mu\nu} = \left(\frac{e}{m_0 c}\right)^2 (B^2 - \mathcal{E}^2) \tag{3.4}$$

$$\omega\cdot\omega + \omega^*\cdot\omega^* = -\omega^2 I \tag{3.5}$$

$$\omega^*\cdot\omega = \omega\cdot\omega^* = i(\vec{\mathcal{E}}\cdot\vec{B})\left(\frac{e}{m_0 c}\right)^2 I = [\operatorname{sgn}(\vec{\mathcal{E}}\cdot\vec{B})](\det \omega) I \tag{3.6}$$

Below, we obtain two orthogonal projection operators analogous to those of the previous section

$$\Pi_\perp + \Pi_\parallel = 1 \quad , \qquad \Pi_\perp \cdot \Pi_\parallel = \Pi_\parallel \cdot \Pi_\perp = 0 \tag{3.7}$$

which separate u into two orthogonal parts

$$u_\perp \equiv \Pi_\perp \cdot u \quad , \qquad u_\parallel = u - u_\perp \tag{3.8}$$

$$u_{\parallel} \cdot u_{\perp} = u_{\perp} \cdot u_{\parallel} = 0 \tag{3.9}$$

in such a way that $u_{\parallel}$ represents the generalization of the standard guiding center velocity. The instructive special case $\vec{\mathcal{E}} \cdot \vec{B} = 0$ is discussed in Appendix F.

Consider the operator

$$\prod(\omega) = \frac{-\frac{1}{\omega^2}\,\omega\cdot\omega+\lambda I}{1 + 2\lambda} \tag{3.10}$$

where λ is a scalar determined from the condition

$$\prod(\underline{\omega}) \cdot \prod(\underline{\omega}^*) = \prod(\underline{\omega}^*) \cdot \prod(\underline{\omega}) = 0 \tag{3.11}$$

Notice that we have normalized so that

$$\prod(\omega) + \prod(\underline{\omega}^*) = 1 \tag{3.12}$$

From Eqs. (3.10) and (3.11) we find that λ satisfies a quadratic. We choose the root

$$\lambda = \frac{-1 + \sqrt{1 - \frac{\det \omega}{\omega^4}}}{2} = \frac{1}{2}\left[-1 + \sqrt{1 + 4\left(\frac{\vec{\mathcal{E}}\cdot\vec{B}}{B^2-\mathcal{E}^2}\right)^2}\,\right] \tag{3.13}$$

to satisfy the condition that $\lambda \to 0$ as $\vec{\mathcal{E}}\cdot\vec{B} \to 0$ since in this case (3.6) yields the projection property, i.e., (3.7). Using Eq. (3.13), we obtain

$$\prod(\underline{\omega}) = \frac{-\omega\cdot\omega + \frac{1}{2}\left[\omega^2 + \frac{\omega^2}{|\omega^2|}\sqrt{\frac{\omega^2}{2} - 4\det\omega}\,\right]I}{\frac{\omega^2}{|\omega^2|}\sqrt{\frac{\omega^2}{2} - 4\det\omega}} \tag{3.14}$$

The useful matrix properties of $\prod(\underline{\omega})$ are given in Appendix G. We note that if $\vec{\mathcal{E}}\cdot\vec{B} \neq 0$

$$\lim_{B^2-\mathcal{E}^2\to 0^+} \prod(\underline{\omega}) = \lim_{B^2-\mathcal{E}^2\to 0^-} \prod(\underline{\omega}^*) \tag{3.15}$$

$$\lim_{B^2-\mathcal{E}^2\to 0^-} \prod(\underline{\omega}) = \lim_{B^2-\mathcal{E}^2\to 0^+} \prod(\underline{\omega}^*) \tag{3.16}$$

Therefore, we define the projection operators, $\prod_\perp$ and $\prod_\parallel$, by

$$\prod_\perp = \begin{cases} \prod(\underline{\omega}) B^2 \geq \mathcal{E}^2 \\ \prod(\underline{\omega}^*) B^2 \leq \mathcal{E}^2 \end{cases} , \qquad \prod_\parallel = \begin{cases} \prod(\underline{\omega}^*) B^2 \geq \mathcal{E}^2 \\ \prod(\underline{\omega}) B^2 \leq \mathcal{E}^2 \end{cases} \tag{3.17}$$

Then $\prod_\perp$ and $\prod_\parallel$ are continuous at $B^2-\mathcal{E}^2=0$ and they have the desired properties defined by Eqs. (3.10). From (3.1), we see that $u_\perp$ and $u_\parallel$ obey the same equation of motion as $u(\tau)$ does. Therefore, if $u_\perp(0)=0$, then $u_\perp(\tau)=0$ for all τ . Thus, the guiding center represents a particle for which $u_\perp(0)=0$. Since $u_\perp$ is space-like (see (4.10)), it cannot be directly related to the motion of a particle. Furthermore, $u_\parallel^2$ and $u_\perp^2$ are Lorentz and gauge-invariant constants of the motion.

By differentiating Eq. (3.1)

$$\frac{d^2u_\perp}{d\tau^2} = \omega\cdot\omega\cdot u_\perp \quad , \qquad \frac{d^2u_\parallel}{d\tau^2} = \omega\cdot\omega\cdot u_\parallel \tag{3.18}$$

But, from Eq. (3.10)

$$\omega\cdot\omega = \omega^2[\lambda I - (1+2\lambda)\prod(\underline{\omega})] \tag{3.19}$$

Therefore

$$\frac{d^2u_\perp}{d\tau^2} = -\omega_\perp^2 u^\perp \qquad \omega_\perp^2 = \begin{cases} \omega^2(1+\lambda) & B^2 > \mathcal{E}^2 \\ \omega^2\lambda & \mathcal{E}^2 > B^2 \end{cases} \tag{3.20}$$

$$\frac{d^2u_\parallel}{d\tau^2} = \omega_\parallel^2 u_\parallel \qquad \omega_\parallel^2 = \begin{cases} \omega^2\lambda & B^2 > \mathcal{E}^2 \\ \omega^2(1+\lambda) & \mathcal{E}^2 > B^2 \end{cases} \tag{3.21}$$

The solutions of Eqs. (3.20) and (3.21) are

$$u_\perp(\tau) = \left[\frac{\omega}{\omega_\perp}\sin\omega_\perp\tau + I\cos\omega_\perp\tau\right]\cdot\prod_\perp\cdot u(0) \tag{3.22}$$

$$u_\parallel(\tau) = \left[\frac{\omega}{\omega_\parallel}\sinh\omega_\parallel\tau + I\cosh\omega_\parallel\tau\right]\cdot\prod_\parallel\cdot u(0) \tag{3.23}$$

Therefore

$$e^{\omega\tau} = \left(\frac{\omega}{\omega_\perp}\sin\omega_\perp\tau + I\cos\omega_\perp\tau\right)\prod_\perp + \left(\frac{\omega}{\omega_\parallel}\sinh\omega_\parallel\tau + I\cosh\omega_\parallel\tau\right)\prod_\parallel \tag{3.24}$$

Integrating the resulting expression for the velocity, we obtain

$$x - x(0) = \left[I\left(\frac{\sin\omega_\perp\tau}{\omega_\perp}\right) + \frac{\omega^*}{\sqrt{\det\omega}}(\cos\omega_\perp\tau - 1)\right]\cdot u_\perp(0) +$$

$$+ \left[I\left(\frac{\sinh\omega_\parallel\tau}{\omega_\parallel}\right) + \frac{\omega}{\sqrt{\det\omega}}(\cosh\omega_\parallel\tau - 1)\right]\cdot u_\parallel(0) \tag{3.25}$$

Below, we use this expression to study the motion of the generalized guiding center.

4. STANDARD AND SINGULAR FIELD CONFIGURATIONS

The possible field configurations are given in Table 1. Below, we show that only two configurations, the "standard" and the "singular," need separate treatment, all others being related to the standard configuration either as special cases or by an explicitly given Lorentz transformation.

$\vec{\mathcal{E}} \cdot \vec{B} = 0$	$B^2 - \mathcal{E}^2 = 0$ (a) $B^2 - \mathcal{E}^2 > 0$ (b) $B^2 - \mathcal{E}^2 < 0$ (c)
$\vec{\mathcal{E}} \cdot \vec{B} \neq 0$	$B^2 - \mathcal{E}^2 = 0$ (d) $B^2 - \mathcal{E}^2 > 0$ (e) $B^2 - \mathcal{E}^2 < 0$ (f)

Table 1. Possible Field Configurations

(a) The operators constructed in Section 3 are singular; thus, we call this the "singular" configuration. We give the solution of the particle equation of motion for this configuration separately.

(b) This is the familiar crossed-field case where the magnetic field dominates. The standard technique which is used to obtain solutions of the particle equation of motion in this case is to consider the motion in a Lorentz frame in which $\mathcal{E}^2 = 0$. The motion in the original frame is obtained by merely adding the constant transformation velocity to the solution: $\vec{u} = (c/B^2)(\vec{\mathcal{E}} \times \vec{B})$. Thus, by studying the motion of the particle in a uniform B field, we actually study the entire class of field configurations.

(c) Using similar reasoning, we choose the pure $\vec{\mathcal{E}}$-field as the prototype for this class.

(d,e,f) We will see that when $\vec{\mathcal{E}}$ and $\vec{B}$ are parallel, $\prod_{\parallel}$ $\prod_{\perp}$ reduce to particularly simple diagonal forms which are independent of the magnitudes of the field strengths. Furthermore, the parallel field configuration can be reached from any nonperpendicular configuration through a Lorentz transformation given in Appendix B . Therefore, if we adopt the parallel field case as the prototype of each of classes d, e, and f , we may study all three of these in one example. Furthermore, since $B^2 - \mathcal{E}^2$ can take on any value in this example, we can also study the special cases, $B^2 = 0$ or $\mathcal{E}^2 = 0$, but these are the pure field cases which we have adopted as the prototypes of b and c . Thus, the prototypes of all classes except case a can be studied through one example in which the $\prod$'s take on very simple forms.

We now give explicit solutions of the equations of motion in the standard and in the singular configurations.

a. <u>Standard Configuration</u>

In the standard field configuration (S.C.) we let the $\vec{\mathcal{E}}$- and $\vec{B}$-fields be parallel and in the z direction. Then $\prod_{\perp}$ and $\prod_{\parallel}$ are given by

$$\prod_{\perp} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \prod_{\parallel} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{(S.C.)} \quad (4.1)$$

Thus,

$$u_{\perp} = (\gamma v_1, \gamma v_2, 0, 0) \quad , \qquad u_{\parallel} = (0, 0, \gamma v_3; i\gamma c) \quad , \quad \text{(S.C.)} \quad (4.2)$$

In the standard configuration, $\vec{u}_{\parallel} = \gamma \vec{v}_{\parallel}$ represents the motion of the particle along the magnetic field. Henceforth, we will call $u_{\parallel}$ the world velocity of the generalized guiding center.

From Eq. (4.2) we see that

$$u_{\perp}^2 \geq 0 \qquad (4.3)$$

but we always have $-c^2 = u_{\perp}^2 + u_{\parallel}^2$ and therefore

$$u_{\parallel}^2 \leq -c^2 \qquad (4.4)$$

Thus, $u_{\parallel}$ is time-like in the standard configuration and $u_{\perp}$ is

space-like. By the discussion of the previous subsection, these properties hold also in any configuration. From Eqs. (4.2), if $x(\tau)$ is the 4-space position of the particle, then the position of the guiding center is given by

$$x_{\parallel} = (0,0,x_3(\tau),x_4(\tau)) \tag{4.5}$$

and the position of the guiding center "complement" is given by

$$x_{\perp} = (x_1(\tau),x_2(\tau),0,0) \tag{4.6}$$

With the fields in the x_3 direction, the vector components of Eq. (3.25) are given by

$$\Delta x_1 = x_1(\tau) - x_1(0) = u_1(0)\left(\frac{\sin\omega_B\tau}{\omega_B}\right) - u_2(0)\left(\frac{\cos\omega_B\tau-1}{\omega_B}\right) \tag{4.7}$$

$$\Delta x_2 = x_2(\tau) - x_2(0) = u_2(0)\left(\frac{\sin\omega_B\tau}{\omega_B}\right) + u_1(0)\left(\frac{\cos\omega_B\tau-1}{\omega_B}\right) \tag{4.8}$$

$$\Delta x_3 = x_3(\tau) - x_3(0) = u_3(0)\left(\frac{\sinh\omega_{\mathcal{E}}\tau}{\omega_{\mathcal{E}}}\right) + c\gamma(0)\left(\frac{\cosh\omega_{\mathcal{E}}\tau-1}{\omega_{\mathcal{E}}}\right) \tag{4.9}$$

$$t = \gamma(0)\left(\frac{\sinh\omega_{\mathcal{E}}\tau}{\omega_{\mathcal{E}}}\right) + \left(\frac{u_3(0)}{c}\right)\left(\frac{\cosh\omega_{\mathcal{E}}\tau-1}{\omega_{\mathcal{E}}}\right) \tag{4.10}$$

where

$$\omega_{\mathcal{E}}^2 = \left(\frac{e\mathcal{E}}{m_0c}\right)^2 , \qquad \omega_B^2 = \left(\frac{eB}{m_0c}\right)^2 \tag{4.11}$$

and where we have set $t(0) = 0$. This result is valid for any $\mathcal{E}$ and B . In addition, without a loss of generality, we may set $u_3(0) = 0$, since a Lorentz transformation parallel to the fields does not affect the fields at all.

In order to write Δx_1 , Δx_2 , and Δx_3 as functions of t , it is convenient to introduce the new variables

$$x^{(+)}(\tau) \equiv \Delta x_3 + i\Delta x_4 , \qquad x^{(-)}(\tau) \equiv \Delta x_3 - i\Delta x_4 \tag{4.12}$$

where $\Delta x_4 = ict$. Then

$$u^{(+)} \equiv u_3 + iu_4 \quad , \quad u^{(-)} \equiv u_3 - iu_4 \tag{4.13}$$

Now, Eqs. (4.12) are equivalent to

$$x^{(+)}(\tau) = \left(\frac{u^{(+)}(0)}{\omega_\mathcal{E}}\right)\left[1 - e^{-\omega_\mathcal{E}\tau}\right] , \; x^{(-)}(\tau) = \left(\frac{u^{(-)}(0)}{\omega_\mathcal{E}}\right)\left[e^{\omega_\mathcal{E}\tau} - 1\right] \tag{4.14}$$

Equations (4.14) may be solved for $\omega_\mathcal{E}\tau$. We obtain

$$\begin{aligned} \omega_\mathcal{E}\tau &= \ell n\left[\left(\frac{\omega_\mathcal{E}}{u^{(-)}(0)}\right)x^{(-)} + 1\right] \\ &= -\ell n\left[\left(\frac{\omega_\mathcal{E}}{u^{(+)}(0)}\right)x^{(+)} - 1\right] \end{aligned} \tag{4.15}$$

Upon eliminating $\omega_\mathcal{E}\tau$, we find an implicit relationship between Δx_3 and t which can be inverted to

$$\Delta x_3(t) = \left(\frac{c\gamma(0)}{\omega_\mathcal{E}}\right)\left[\left(1 + \left(\frac{\omega_\mathcal{E} t}{\gamma(0)}\right)^2\right)^{1/2} - 1\right] \tag{4.16}$$

Equation (4.16) is a general statement of the 3-position of the guiding center in the standard configuration as a function of time.

With Eq. (4.16) and $x^{(-)} = \Delta x_3 + ct$, we may obtain an explicit expression for $\tau(t)$

$$\omega_B\tau = \left(\frac{\omega_B}{\omega_\mathcal{E}}\right)\ell n\left[\left(\frac{\omega_\mathcal{E} t}{\gamma(0)}\right) + \left(1 + \left(\frac{\omega_\mathcal{E} t}{\gamma(0)}\right)^2\right)^{1/2}\right] \tag{4.17}$$

This last expression may be substituted directly into Eqs. (4.10) and (4.11) to obtain $\Delta x_1(t)$ and $\Delta x_2(t)$.

For $(\omega_{\mathcal{E}} t/\gamma(0)) \ll 1$, Eq. (4.17) is given by

$$\omega_B \tau \simeq \left(\frac{\omega_B}{\omega_{\mathcal{E}}}\right) \ell n \left[1 + \frac{\omega_{\mathcal{E}} t}{\gamma(0)}\right] \approx \frac{\omega_B t}{\gamma(0)} \tag{4.18}$$

With this expression, we find

$$\begin{aligned} \Delta x_1 &\simeq v_1(0)\left(\frac{\sin\,(\omega_B/\gamma(0))t}{\omega_B/\gamma(0)}\right) - v_2(0)\left(\frac{\cos\,(\omega_B/\gamma(0))t - 1}{\omega_B/\gamma(0)}\right) \\ \Delta x_2 &\simeq v_2(0)\left(\frac{\sin\,(\omega_B/\gamma(0))t}{\omega_B/\gamma(0)}\right) + v_1(0)\left(\frac{\cos\,(\omega_B/\gamma(0))t - 1}{\omega_B/\gamma(0)}\right) \end{aligned} \tag{4.19}$$

and from Eq. (4.16)

$$\Delta x_3 \simeq \frac{1}{2}\left(\frac{\omega_{\mathcal{E}} c}{\gamma(0)}\right) t^2 \tag{4.20}$$

Note that

$$v_3 \equiv \frac{d}{dt}(\Delta x_3) \simeq c\left(\frac{\omega_{\mathcal{E}} t}{\gamma(0)}\right) \tag{4.21}$$

Thus, the condition that $(\omega_{\mathcal{E}} t/\gamma(0)) \ll 1$ is equivalent to the statement that the motion of the particle along the fields is subrelativistic. Either t is small enough or $\vec{\mathcal{E}}$ weak enough such that the particle is not appreciably accelerated along the fields. The motion perpendicular to the fields need not be subrelativistic, however.

Equations (4.12) and (4.13) are exact when $\vec{\mathcal{E}}$ and $\omega_{\mathcal{E}}$

are equal to zero. The nonrelativistic gyration frequency is corrected to include the relativistic mass; i.e., the relativistic gyration frequency is

$$\frac{\omega_B}{\gamma} = \frac{eB}{\gamma m_0 c} = \frac{eB}{mc} \tag{4.22}$$

If $\vec{\mathcal{E}} \neq 0$, the particle is initially accelerated uniformly along the electric field and the motion perpendicular to the fields is unaffected.

When $(\omega_{\mathcal{E}} t/\gamma(0)) \gg 1$, Eqs. (4.1) and (4.17) are given by

$$\Delta x_3 \simeq ct \tag{4.23}$$

and

$$\omega_B \tau \simeq \frac{\omega_B}{\omega_{\mathcal{E}}} \ln \frac{2\omega_{\mathcal{E}} t}{\gamma(0)} \tag{4.24}$$

Thus, in 3-space, the guiding center moves along the electric field with $v_3 \simeq c$ and the guiding center complement still gyrates about the guiding center but at a much slower rate than in the subrelativistic regime.

b. Explicit Time Dependence for the Singular Configuration

Let the field components be given by

$$\vec{\mathcal{E}} = (B,0,0) \quad \text{and} \quad \vec{B} = (0,B,0) \tag{4.25}$$

The components of Eq. (3.1) are given by

$$\frac{d}{dt} u_1 = -\omega_B u_3 - i\omega_B u_4 \quad , \quad \frac{d}{dt} u_2 = 0 \tag{4.26}$$

$$\frac{d}{dt} u_3 = \omega_B u_1 \quad , \quad \frac{d}{dt} u_4 = i\omega_B u_1 \tag{4.27}$$

For initial conditions, we choose without loss of generality

$$t(\tau=0) = 0 \tag{4.28}$$

$$v_1(0) = v_2(0) = v_3(0) = 0 \tag{4.29}$$

With these initial conditions, we find

$$u_1 = c(\omega_B\tau) \quad , \quad u_2 = 0 \tag{4.30}$$

$$u_3 = (1/2)c(\omega_B\tau)^2 \quad , \quad u_4 = ic + (1/2)c(\omega_B\tau)^2 \tag{4.31}$$

From Eq. (4.31) for $u_4 = i\gamma c$, we find

$$\gamma = 1 + (1/2)(\omega_B\tau)^2 \tag{4.32}$$

Thus, the equation $dt/d\tau = \gamma$ can be integrated directly to find the cubic equation

$$t = \tau\left[1 + \tfrac{1}{6}(\omega_B\tau)^2\right] \tag{4.33}$$

Equation (4.33) can be inverted to find

$$\tau(t) = (3\omega_B t)^{1/3}\left\{\left[\left(1+ \frac{8}{(3\omega_B t)^2}\right)^{1/2} + 1\right]^{1/3} - \left[\left(1+ \frac{8}{(3\omega_B t)^2}\right)^{1/2} - 1\right]^{1/3}\right\} \tag{4.34}$$

Equations (4.30) and (4.31) can be rewritten for the three velocities. We obtain

$$\frac{v_1}{c} = \frac{\omega_B\tau}{1 + \frac{1}{2}(\omega_B\tau)^2} \quad , \quad \frac{v_2}{c} = 0 \quad , \quad \frac{v_3}{c} = \frac{\omega_B\tau}{2}\,\frac{v_1}{c} \tag{4.35}$$

Equations (4.34) and (4.35) give the solution for $\vec{v}(t)$.

For $\omega_B t$, Eqs. (4.34) and (4.35) reduce to the nonrelativistic result

$$\tau \approx t \quad , \quad \frac{v_1}{c} \simeq \omega_B t \quad , \quad \frac{v_3}{c} \simeq \frac{1}{2}(\omega_B t)^2 \tag{4.36}$$

The particle is initially accelerated uniformly along the electric field and slowly gains a velocity component in the z-direction orthogonal to both the fields. After the particle

is accelerated for a long enough time, we have $\omega_B t \gg 1$
Then

$$\omega_B \tau \simeq (6\omega_B t)^{1/3} \tag{4.37}$$

and

$$\frac{v_1}{c} \simeq \frac{2}{(6\omega_B t)^{1/3}} \qquad \frac{v_3}{c} \simeq 1 \tag{4.38}$$

5. CONCLUSION

The operator technique developed here has been shown to be very useful in describing the relativistic guiding center and also the general motion of a charged particle in an arbitrary uniform field configuration. The relative simplicity of the technique leads us to believe that it may also prove useful when the fields are not uniform and stationary. We shall discuss the application of our projection operators to the motion of a particle in fields which have small variations about the uniform configurations elsewhere. Since the positions of the guiding center and its complement are given in terms of the particle's position by simple contact transformations in space-time, it is not at all clear that the usual perturbation techniques will be necessary; exact canonical equations of motion for the guiding center and its complement may be obtainable.

Our main results are summarized in Table 2 where the pure magnetic field is compared with the general case.

$\frac{d\vec{v}}{dt} = \underline{\Omega} \cdot \vec{v}$ $\vec{v} = \vec{v}_\perp + \vec{v}_\parallel$ $\vec{v}_\perp = \underline{\mathcal{P}}_\perp \cdot \vec{v}$	$\frac{du}{d\tau} = \underline{\omega} \cdot u$ $\vec{u} = u_\perp + u_\parallel$ $u_\perp = \underline{\Pi}_\perp \cdot u$
$\underline{\mathcal{P}}_\perp = -\frac{1}{\omega_B^2} \underline{\Omega} \cdot \underline{\Omega}$ $\omega_B^2 = B^2\left(\frac{e}{m_0 c}\right)^2$	$\underline{\Pi}(\underline{\omega}) = \left[-\frac{1}{\omega^2} \underline{\omega} \cdot \underline{\omega} + \lambda I\right] \frac{1}{1+2\lambda}$ $\underline{\Pi}_\perp = \begin{cases} \underline{\Pi}(\underline{\omega}) & B^2 > \mathcal{E}^2 \\ \underline{\Pi}(\underline{\omega}^*) & \mathcal{E}^2 > B^2 \end{cases}$ $\omega^2 = (B^2 - \mathcal{E}^2)\left(\frac{e}{m_0 c}\right)^2$ $\lambda = \left\{\frac{1}{2} - 1 + \sqrt{1 + 4\left(\frac{\mathcal{E} \cdot B}{B^2 - \mathcal{E}^2}\right)^2}\right\} \geq 0$
Gyr $\frac{d^2 \vec{v}_\perp}{dt^2} + \omega_B^2 \vec{v}_\perp = 0$ G.C. $\frac{d^2 \vec{v}_\parallel}{dt^2} = 0$	$\frac{d^2 u_\perp}{d\tau^2} + \lvert\omega^2\rvert \lambda_\perp u_\perp = 0 \qquad u_\perp \in S$ $\frac{d^2 u_\parallel}{d\tau^2} - \lvert\omega^2\rvert \lambda_\parallel u_\parallel = 0 \qquad u_\parallel \in T^+$ $\lambda_\perp = \begin{cases} 1 + \lambda & B^2 > \mathcal{E}^2 \\ \lambda & \mathcal{E}^2 > B^2 \end{cases} \qquad \lambda_\perp + \lambda_\parallel = 1 + 2\lambda$

Table 2. Comparison of the Pure Magnetic Field with the General Case

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APPENDIX F

Note that

$$\tilde{\omega}^1 = -\frac{\omega \cdot \omega}{\omega^2} , \quad \tilde{\omega}^2 \doteq -\frac{\omega^* \cdot \omega^*}{\omega^2} \tag{F1}$$

form a complete but nonorthogonal set since, from Eq. (3.9),

$$\tilde{\omega}^2 \cdot \tilde{\omega}^1 = \tilde{\omega}^1 \cdot \tilde{\omega}^2 = \frac{\det \omega}{\omega^4} I \tag{F2}$$

Thus $\tilde{\omega}^2$ and $\tilde{\omega}^1$ play the role of the projection operators if $\vec{\mathcal{E}} \cdot \vec{B} = 0$. Furthermore, from Appendix G,

$$\tilde{\omega}^2 \cdot \omega^* = \omega^* \quad , \quad \tilde{\omega}^2 \cdot \omega = 0 \quad , \quad (\vec{\mathcal{E}} \cdot \vec{B} = 0) \tag{F3}$$

$$\tilde{\omega}^1 \cdot \omega = \omega \quad , \quad \tilde{\omega}^1 \cdot \omega^* = 0 \quad , \quad (\vec{\mathcal{E}} \cdot \vec{B} = 0) \tag{F4}$$

Introducing

$$u^1 = \tilde{\omega}^1 \cdot u \quad , \quad u^2 = \tilde{\omega}^2 \cdot u \tag{F5}$$

we find

$$\frac{du^1}{d\tau} = \frac{du}{d\tau} \quad , \quad \frac{du^2}{d\tau} = 0 \tag{F6}$$

The vector u^2 represents the constant velocity of the relativistic guiding center when $\vec{\mathcal{E}} \cdot \vec{B} = 0$. In fact, the expression

$$u^2 = u - u^1 = u - \tilde{\omega}^1 \cdot u = u + \left(\frac{\omega \cdot \omega}{\omega^2}\right) \cdot u = u + \frac{1}{\omega^2}\, \omega \cdot \frac{du}{d\tau} \tag{F7}$$

integrates to

$$X^* = x + \frac{\omega \cdot u}{\omega^2} + C \tag{F8}$$

whose 3-part with $\vec{\mathcal{E}} = 0$ coincides with the position of the guiding center (2.14), but $\mathcal{E} \cdot B = 0$.

APPENDIX G. USEFUL MATRIX PROPERTIES

Using (3.11) and (3.12), we find

$$\tilde{\omega}^1 \cdot \tilde{\omega}^1 = \tilde{\omega}^1 - \frac{\det \omega}{\omega^4} I \;, \quad \tilde{\omega}^2 \cdot \tilde{\omega}^2 = \tilde{\omega}^2 - \frac{\det \omega}{\omega^4} I \qquad \text{(G1)}$$

Also, notice that

$$\tilde{\omega}^2 \cdot \omega^* = \omega^* \cdot \tilde{\omega}^2 = \omega^* + \sigma\sqrt{\frac{\det \omega}{\omega^4}}\,\omega$$

$$\tilde{\omega}^2 \cdot \omega = \omega \cdot \tilde{\omega}^2 = -\sigma\sqrt{\frac{\det \omega}{\omega^4}}\,\omega^* \qquad \text{(G2)}$$

and

$$\tilde{\omega}^1 \cdot \omega = \omega \cdot \tilde{\omega}^1 = \omega + \sigma\sqrt{\frac{\det \omega}{\omega^4}}\,\omega^*$$

$$\tilde{\omega}^1 \cdot \omega^* = \omega^* \cdot \tilde{\omega}^1 = -\sigma\sqrt{\frac{\det \omega}{\omega^4}}\,\omega \qquad \text{(G3)}$$

In Eqs. (G1) and (G2) we have introduced the notation

$$\sigma = \operatorname{sgn}\,(\vec{\mathcal{E}} \cdot \vec{B}\,[B^2 - \mathcal{E}^2]) \qquad \text{(G4)}$$

With these relationships, we may go in a straightforward manner to obtain

$$\prod^2 \cdot \omega = \omega \cdot \prod^2 = \frac{-\sigma\sqrt{\frac{\det \omega}{\omega^4}}\,\omega^* + \lambda\omega}{1 + 2\lambda} \qquad \text{(G5)}$$

$$\prod^2 \cdot \omega^* = \omega^* \cdot \prod^2 = \frac{(1+\lambda)\omega^* + \sigma\sqrt{\frac{\det \omega}{\omega^4}}\,\omega}{1 + 2\lambda} \qquad \text{(G6)}$$

$$\prod^1 \cdot \omega = \omega \cdot \prod^1 = \frac{(1+\lambda)\omega + \sigma\sqrt{\frac{\det \omega}{\omega^4}}\,\omega^*}{1 + 2\lambda} \qquad \text{(G7)}$$

and

$$\prod^{1} \cdot \omega^{*} = \omega^{*} \cdot \prod^{1} = \frac{-\ \sigma\sqrt{\frac{\det \omega}{\omega^{4}}}\ \omega + \lambda\omega^{*}}{1 + 2\lambda} \tag{G8}$$

We shall use Eqs. (G5) through (G8) in our construction of the equations of motion of the projected world velocities.

APPENDIX H

Given an arbitrary field configuration $(\mathcal{E},B,\theta)$ where θ is the angle between $\mathcal{E}$ and B, we can find the equivalent standard configuration $(\mathcal{E}', B', \theta' = 0)$. The magnitudes of the standard fields are

$$B'^2 = \frac{1}{2}(B^2 - \mathcal{E}^2)\left[1 + \frac{\sqrt{(B^2-\mathcal{E}^2)^2 + 4(\mathcal{E}B)^2\cos^2\theta}}{B^2 - \mathcal{E}^2}\right] \qquad \text{(H1)}$$

$$\mathcal{E}'^2 = \frac{1}{2}(\mathcal{E}^2 - B^2)\left[1 + \frac{\sqrt{(B^2-\mathcal{E}^2)^2 + 4(\mathcal{E}B)^2\cos^2\theta}}{\mathcal{E}^2 - B^2}\right] \qquad \text{(H2)}$$

The velocity which accomplishes the transformation from the arbitrary to the standard configuration is given by

$$\vec{V} = \alpha(\vec{\mathcal{E}} \times \vec{B})c \qquad \text{(H3)}$$

where

$$\alpha = \frac{(B^2+\mathcal{E}^2) - \sqrt{(B^2-\mathcal{E}^2)^2 + 4(\mathcal{E}B)^2\cos^2\theta}}{2(\mathcal{E}B)^2\sin^2\theta} \qquad \text{(H4)}$$

Thus, for arbitrary $(\mathcal{E},B,\theta)$ it is possible using the proper Lorentz transformation to obtain $(\mathcal{E}',B',\theta)$ with $\vec{\mathcal{E}}'$ and $\vec{B}'$ in the z'-direction. Then $\prod_\perp$ and $\prod_\parallel$ take on simple forms which are independent of the magnitudes $\mathcal{E}'$ and B'

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